British Journal of Mathematics & Computer Science
20(4): 1-13, 2017; Article no.BJMCS.31193
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

General Version of Gauss-type Proximal Point Method and Its Uniform Convergence Analysis for Metrically Regular Mappings

M. A. Alom[1,2], M. H. Rashid[1*] and K. K. Dey[1]

[1]*Department of Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh.*
[2]*Department of Mathematics, Khulna University of Engineering and Technology, Khulna-9203, Bangladesh.*

Authors' contributions

This work was carried out in collaboration between all authors. Author MAA gathered the initial data, managed the analysis of the study and drafting the article. Author MHR designed the study, created the idea to develop a mathematical algorithm and designed a methodology for programming and interpreted the results. Author KKD helped the literature searching. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/BJMCS/2017/31193
Editor(s):
(1) Serkan Araci, Hasan Kalyoncu University, Turkey.
(2) Tian-Xiao He, Department of Mathematics and Computer Science, Illinois Wesleyan University, USA.
Reviewers:
(1) Abdullah Sonmezoglu, Bozok University, Turkey.
(2) Rashmi Awad, Devi Ahilya University, Indore, India.
(3) M. Thiyagarajan, NIET Anna University, India.
(4) Huseyin Dal, Sakarya University, Turkey.
Complete Peer review History: http://www.sciencedomain.org/review-history/17880

Received: ***25th December 2016***
Accepted: ***3rd February 2017***
Published: ***17th February 2017***

Original Research Article

Abstract

We study the uniform convergence of the general version of Gauss-type proximal point algorithm (GG-PPA), introduced by Alom et al. [1], for solving the parametric generalized equations $y \in T(x)$, where $T : X \rightrightarrows 2^Y$ is a set-valued mapping with locally closed graph, y is a parameter, and X and Y are Banach spaces. In particular, we establish the uniform convergence of the GG-PPA by considering a sequence of Lipschitz continuous functions $g_k : X \to Y$ with $g_k(0) = 0$ and positive Lipschitz constants λ_k in the sense that it is stable under small perturbations when T is metrically regular at a given point. In addition, we give a numerical example to justify the uniform convergence of the GG-PPA.

**Corresponding author: E-mail: harun_math@ru.ac.bd*

Keywords: *Set-valued mappings; metrically regular mappings; lipschitz-like mappings; semi-local convergence; uniform convergence.*

2010 Mathematics Subject Classification: *47H04, 49J53, 65K10.*

1 Introduction

Let X and Y be Banach spaces and Ω be an open subset of X. Let $T : X \rightrightarrows 2^Y$ be a set valued mapping that have locally closed graph. We deal with the problem of approximating a point $x \in \Omega$ satisfying the generalized equation

$$y \in T(x), \tag{1.1}$$

where y is a parameter. Robinson [2][3] first introduced the generalized equation (1.1) for $y = 0$ as a general mechanism for describing, analyzing, and solving different problems in a unified way. This kind of problems have been reviewed broadly. Various examples are system of inequalities, variational inequalities, linear and nonlinear complementary problems, system of nonlinear equations, equilibrium problems, etc.; see in [1][3][4][5].

The proximal point algorithm (PPA) is one of the most important methods for solving (1.1) in the case $y = 0$, which is defined as follows:

$$0 \in \lambda_k(x_{k+1} - x_k) + T(x_{k+1}), \quad \text{for each } k = 0, 1, 2, \dots, \tag{1.2}$$

where $\{\lambda_k\} \subseteq (0, \lambda)$ is a sequence of scalars. This PPA, whose origin can be found in the works of Martinet [6] for variational inequalities, has been further studied and extended in [7][8][9][10][11] to a more general setting, including convex programs, convex-concave saddle point problems and variational inequality problems. Rockafellar [11] throughly analyzed the PPA in the general structure of maximal monotone inclusions. When $y = 0$, Aragón Artacho et al. [12] have been introduced the general version of proximal point algorithm (GPPA) for solving (1.1) by choosing a sequence of functions $g_k : X \to Y$ with $g_k(0) = 0$ which are Lipschitz continuous in a neighborhood O of the origin with Lipschitz constants λ_k for each k and established the linear and super-linear convergence results under certain conditions. Let $x \in X$ satisfying $y \in T(x)$. The subset of X, denoted by $\mathcal{D}^k(x)$, is defined by

$$\mathcal{D}^k(x) := \Big\{ d \in X : y \in g_k(d) + T(x + d) \Big\}. \tag{1.3}$$

Moreover, Aragón Artacho and Geoffroy [13] have been presented the following GPPA for solving (1.1) and proved the uniform convergence of GPPA:

Algorithm 1 (General version of proximal point algorithm (GPPA))

Step 0. Let $x_0 \in X$, $\lambda \in (0, \infty)$ and put $k := 0$.
Step 1. If $0 \in \mathcal{D}^k(x_k)$, then stop; otherwise, go to Step 2.
Step 2. Put $\{\lambda_k\} \subseteq (0, \lambda)$, $g_k(0) = 0$. If $0 \notin \mathcal{D}^k(x_k)$, choose d_k such that $d_k \in \mathcal{D}^k(x_k)$.
Step 3. Write $x_{k+1} := x_k + d_k$.
Step 4. Put $k = k + 1$ and go to Step 1.

Basically, the sequences generated by Algorithm 1 are not uniquely defined and not every generated sequence is convergent. Under certain conditions, Aragón Artacho and Geoffroy [13] showed that there exists one sequence $\{x_n\}$ generated by Algorithm 1, which is linearly convergent to the solution. Hence in view of mathematical computation, this type of methods are not convenient in practical application. Thus, to overcome this barrier, we propose a method "so called" the general version of Gauss-type proximal point algorithm (GG-PPA) for solving (1.1) as follows:

Algorithm 2 (General version of Gauss-type proximal point algorithm (GG-PPA))

Step 0. Select $\eta \in [1, \infty)$, $x_0 \in X$, $\lambda \in (0, \infty)$ and put $k := 0$.
Step 1. If $0 \in \mathcal{D}^k(x_k)$, then stop; otherwise, go to Step 2.
Step 2. Put $\{\lambda_k\} \subseteq (0, \lambda)$, $g_k(0) = 0$. If $0 \notin \mathcal{D}^k(x_k)$, choose d_k such that $d_k \in \mathcal{D}^k(x_k)$ and $\|d_k\| \leq \eta$ dist $(0, \mathcal{D}^k(x_k))$.
Step 3. Write $x_{k+1} := x_k + d_k$.
Step 4. Put $k = k + 1$ and go to Step 1.

We observe that,

(i) if $\eta = 1$ and $\mathcal{D}^k(x_k)$ is single valued, Algorithm 2 coincides with the Algorithm 1.

(ii) if $g_k(u) = \lambda_k u$, $y = 0$ and $Y = X$ a Banach space, Algorithm 2 is equivalent to the classical Gauss-type proximal point algorithm, which have been introduced by Rashid et al. [9].

(iii) if $g_k(u) = \lambda_k u$ and $Y = X$ a Banach space, Algorithm 2 is equivalent to the classical Gauss-type proximal point algorithm introduced by Rashid in his PhD thesis [4, Algorithm 4.2.1].

(iv) if $y = 0$, Algorithm 2 is equivalent to the algorithm introduced by Alom et al. [1].

The difference between the Algorithm 1 and the Algorithm 2 is that the GG-PPA generates at least one sequence and every generated sequence is convergent, but this does not happen for the Algorithm 1.

The main goal in this paper is to present a kind of convergence of the sequence generated by Algorithm 2, which is uniform in the sense that the attraction region (i.e., the ball in which the initial guess x_0 can be taken arbitrarily) does not depend on small variations of the value of the parameter y near $\bar{y}$ and for such values of y the method finds a solution x of (1.1) whenever T is metrically regular at $(\bar{x}, \bar{y})$.

Many authors have been studied on local convergence analysis about Algorithm 1 in the case $y = 0$, see for examples [2][3][8][11], but there is no semi-local analysis for the Algorithm 1. A large number of participations have been studied on semi-local analysis for the Gauss-Newton method (cf. [14][15][16]). A semi-local convergence analysis for the classical Gauss-type proximal point method were presented by Rashid et al. [9]. A semi-local convergence analysis for the GG-PPA were presented by Alom et al. (cf. [1]). As our best knowledge, there is no study on uniformity of semi-local analysis for the Algorithm 2. Thus, we conclude that the contributions, presented in this study, seem new.

To analyze the uniformity of semi-local convergence of the GG-PPA is the main task in this paper. The main tools in our study are the metric regularity property and Lipschitz-like property for set-valued mappings. Based on the information around the initial point, the main results are the convergence criterion, which provide some sufficient conditions confirming the convergence to a solution of any sequence generated by Algorithm 2. As a consequence, uniform local convergence of the GG-PPA is obtained.

The content of this paper is organized as follows: In section 2, some notations, notions and preliminary results are presented. In Section 3, we consider the GG-PPA which is introduced in section 1. Then utilizing the concept of metric regularity property for the set valued mapping T, we will show the existence of the sequence generated by Algorithm 2 and present the uniform convergence of the GG-PPA. In section 4, we give a numerical example to justify the uniformity of semi-local convergence of the Algorithm 2. The summary of the major results are presented in the last section.

2 Notations and Preliminary Results

In this section, we suppose that X and Y are two real or complex Banach spaces. The closed ball with centered at a and radius r is denoted by $\mathbb{B}_r(a)$. All the norms are denoted by $\|\cdot\|$. For each $x \in X$, the distance from a point x to a set $C \subseteq X$ is defined by $\text{dist}(x, C) := \inf\{\|x - y\| : y \in C\}$, while the excess from the set $B \subseteq X$ to the set C is defined by $e(B, C) := \sup\{\text{dist}(x, C) : x \in B\}$.

Let $F : X \rightrightarrows 2^Y$ be a set-valued mapping. Here $\text{gph}F := \{(x, y) \in X \times Y : y \in F(x)\}$ is the graph of F and $\text{dom}F := \{x \in X : F(x) \neq \emptyset\}$ is the domain of F. The inverse of F is denoted by F^{-1} and is defined by $F^{-1}(y) := \{x \in X : y \in F(x)\}$ for each $y \in Y$.

The concept of metric regularity for set valued mapping in the following definition is taken from [9], and has been studied extensively; see for example [10][17][18][19].

Definition 2.1. *Let $F : X \rightrightarrows Y$ be a set-valued mapping and let $(\bar{x}, \bar{y}) \in \text{gph}F$. Let $r_{\bar{x}} > 0$, $r_{\bar{y}} > 0$ and $\kappa > 0$. Then F is said to be*

(i) *metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant κ if*

$$\text{dist}\big(x, F^{-1}(y)\big) \leq \kappa \ \text{dist}\big(y, F(x)\big) \quad \text{for all } x \in \mathbb{B}_{r_{\bar{x}}}(\bar{x}),\ y \in \mathbb{B}_{r_{\bar{y}}}(\bar{y}).$$

(ii) *metrically regular at $(\bar{x}, \bar{y})$ if there exist constants $r'_{\bar{x}} > 0$, $r'_{\bar{y}} > 0$ and $\kappa' > 0$ such that F is metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r'_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r'_{\bar{y}}}(\bar{y})$ with constant κ'.*

Recall the definition of Lipschitz-like continuity for set-valued mapping from [7]. This notion was introduced by Aubin [20] and has been studied extensively; see for examples [9][18][19].

Definition 2.2. *Let $\Gamma : Y \rightrightarrows 2^X$ be a set-valued mapping and let $(\bar{y}, \bar{x}) \in \text{gph}\Gamma$. Let $r_{\bar{x}} > 0$, $r_{\bar{y}} > 0$ and $M > 0$. Then Γ is said to be Lipschitz-like at $(\bar{y}, \bar{x})$ on $\mathbb{B}_{r_{\bar{y}}}(\bar{y}) \times \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with constant M if the following inequality hold:*

$$e(\Gamma(y_1) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), \Gamma(y_2)) \leq M\|y_1 - y_2\| \quad \text{for any} \quad y_1,\ y_2 \in \mathbb{B}_{r_{\bar{y}}}(\bar{y}).$$

The following result establishes the equivalence relation between metric regularity of a mapping F and the Lipschitz-like continuity of the inverse F^{-1}, which can be seen in [1][10].

Lemma 2.1. *Let $F : X \rightrightarrows 2^Y$ be a set valued mapping and $(\bar{x}, \bar{y}) \in \text{gph}F$. Let $r_{\bar{x}} > 0$, $r_{\bar{y}} > 0$, then F is metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant κ if and only if its inverse $F^{-1} : Y \rightrightarrows 2^X$ is Lipschitz-like at $(\bar{y}, \bar{x})$ on $\mathbb{B}_{r_{\bar{y}}}(\bar{y}) \times \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with constant κ, that is,*

$$e(F^{-1}(y) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), F^{-1}(y')) \leq \kappa\|y - y'\| \quad \text{for all } y, y' \in \mathbb{B}_{r_{\bar{y}}}(\bar{y}).$$

The following concept of Lyusternik-Graves theorem for metrically regular mapping is extracted from [21]. This theorem plays an important role in the theory of metric regularity and proves the stability of metric regularity of a generalized equation under perturbations. We use the following convention: we say that a set $C \subseteq X$ is locally closed at $z \in C$ if there exists $a > 0$ such that the set $C \cap \mathbb{B}_a(z)$ is closed.

Lemma 2.2. *Let $T : X \rightrightarrows 2^Y$ be a set valued mapping with locally closed graph. Let F be metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\kappa > 0$. Let $g : X \to Y$ be a function which is Lipschitz continuous at $\bar{x}$ with Lipschitz constant $\lambda > 0$ such that $\lambda < \kappa^{-1}$. Then the mapping $g+F$ is metrically regular at $(\bar{x}, \bar{y} + g(\bar{x}))$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y} + g(\bar{x}))$ with constant $\dfrac{\kappa}{1-\kappa\lambda}$.*

We end this section with the following lemma, which is known as Banach fixed point lemma, proved in [22].

Lemma 2.3. *Let $\Phi : X \rightrightarrows 2^X$ be a set-valued mapping. Let $\eta_0 \in X$, $r \in (0, \infty)$ and $\alpha \in (0, 1)$ be such that*

$$\operatorname{dist}(\eta_0, \Phi(\eta_0)) < r(1-\alpha) \tag{2.1}$$

and

$$e(\Phi(x_1) \cap \mathbb{B}_r(\eta_0), \Phi(x_2)) \le \alpha \|x_1 - x_2\| \quad \textit{for any} \;\; x_1, x_2 \in \mathbb{B}_r(\eta_0). \tag{2.2}$$

Then Φ has a fixed point in $\mathbb{B}_r(\eta_0)$, that is, there exists $x \in \mathbb{B}_r(\eta_0)$ such that $x \in \Phi(x)$. If Φ is additionally single-valued, then the fixed point of Φ in $\mathbb{B}_r(\eta_0)$ is unique.

3 Uniform Convergence Analysis

This section is devoted to study the uniform convergence of GG-PPA. To do this, let $(\bar{x}, \bar{y}) \in \operatorname{gph}T$ and $r_{\bar{x}} > 0, r_{\bar{y}} > 0$ be such that $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \subseteq \operatorname{dom}T$ and $\mathbb{B}_{r_{\bar{y}}}(\bar{y}) \subseteq T(X)$, the image of T. We assume that T is metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\kappa > 0$ and $\operatorname{gph}T \cap (\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y}))$ is closed.

Indeed, we are intended to prove that whenever T is metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with a constant κ, then for initial guess $\bar{x} \in X$ and for every $y \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$, there is a sequence $\{x_k\}$ generated by Algorithm 2 starting from $\bar{x}$ and converging to a solution x of (1.1) for y. Let $x \in X$ and define a mapping P_x by

$$P_x(\cdot) := g(\cdot - x) + T(\cdot). \tag{3.1}$$

Then we obtain the following equivalence

$$z \in P_x^{-1}(y) \Leftrightarrow y \in g(z-x) + T(z) \;\; \text{for any } z \in X \text{ and } y \in T(z).$$

In particular,

$$\bar{x} \in P_{\bar{x}}^{-1}(\bar{y}) \;\; \text{for each } (\bar{x}, \bar{y}) \in \operatorname{gph}T.$$

Let $(\bar{x}, \bar{y}) \in \operatorname{gph}(g+T)$. Since $g(\cdot - \bar{x})$ is Lipschitz continuous on $O + \bar{x}$, applying Lemma 2.2, we assume that the mapping $P_{\bar{x}}$ is metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\dfrac{\kappa}{1-\kappa\lambda}$. So, by Lemma 2.1, we say that the mapping $P_{\bar{x}}^{-1}$ is Lipschitz-like at $(\bar{y}, \bar{x})$ on $\mathbb{B}_{r_{\bar{y}}}(\bar{y}) \times \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with constant $\dfrac{\kappa}{1-\kappa\lambda}$, that is,

$$e(P_{\bar{x}}^{-1}(y) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), P_{\bar{x}}^{-1}(y')) \le \frac{\kappa}{1-\kappa\lambda}\|y - y'\| \;\; \text{for all } y, y' \in \mathbb{B}_{r_{\bar{y}}}(\bar{y}). \tag{3.2}$$

Suppose that

$$\lim_{x \to \bar{x}} \operatorname{dist}(y, T(x)) = 0. \tag{3.3}$$

Write

$$\bar{r} := \min\Big\{\frac{2r_{\bar{y}} - r_{\bar{x}}\lambda}{2}, \frac{r_{\bar{x}}(1-3\kappa\lambda)}{4\kappa}\Big\}. \tag{3.4}$$

Then

$$\bar{r} > 0 \Leftrightarrow \lambda < \min\Big\{\frac{2r_{\bar{y}}}{r_{\bar{x}}}, \frac{1}{3\kappa}\Big\}. \tag{3.5}$$

The following lemma plays an important role to present the uniform convergence of the GG-PPA, which is due to [9].

Lemma 3.1. *Suppose that $P_{\bar{x}}(\cdot)$ is metrically regular at $(\bar{x},\bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})\times\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\frac{\kappa}{1-\kappa\lambda}$ such that (3.4) and (3.5) are satisfied. Let $x\in\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ and $\mathbb{B}_{r_{\bar{x}}}(0)\subseteq O$. Then $P_x^{-1}(\cdot)$ is Lipschitz-like at $(\bar{y},\bar{x})$ on $\mathbb{B}_{\bar{r}}(\bar{y})\times\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ with constant $\frac{\kappa}{1-3\kappa\lambda}$, that is,*

$$e(P_x^{-1}(y_1)\cap\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x}),P_x^{-1}(y_2))\le\frac{\kappa}{1-3\kappa\lambda}\|y_1-y_2\|\quad \text{for any } y_1,\ y_2\in\mathbb{B}_{\bar{r}}(\bar{y}).$$

To complete our main result, we suppose that a sequence of functions $g_k: X\to Y$ such that $g_k(0)=0$, which are Lipschitz continuous around the origin, the same for all k, with Lipschitz constants λ_k satisfying

$$\lambda:=\sup_k\lambda_k<\frac{1}{6\kappa}. \tag{3.6}$$

We replace g_k instead of g in (3.1), we obtain the mapping $P_x(\cdot)$ as follows:

$$P_x^k(\cdot):=g_k(\cdot-x)+T(\cdot)\quad\text{for each } k=0,1,2,\ldots.$$

and rewrite the equation (3.2) as follows:

$$e(P_{\bar{x}}^{k^{-1}}(y)\cap\mathbb{B}_{r_{\bar{x}}}(\bar{x}),P_{\bar{x}}^{k^{-1}}(y'))\le\frac{\kappa}{1-\kappa\lambda}\|y-y'\|\ \text{ for all } y,y'\in\mathbb{B}_{r_{\bar{y}}}(\bar{y}). \tag{3.7}$$

By Lemma 2.2 and Lemma 2.1 with (3.6), we obtain that the mapping $P_{\bar{x}}^{k^{-1}}(\cdot)$ is Lipschitz-like at $(\bar{y},\bar{x})$ on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})\times\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with constant $\frac{\kappa}{1-\kappa\lambda}$ satisfying (3.7). Thus we have

$$\mathcal{D}^k(x)=\left\{d\in X\colon x+d\in P_x^{k^{-1}}(y)\right\} \tag{3.8}$$

and we obtain the following equivalence

$$z\in P_x^{k^{-1}}(y)\Leftrightarrow y\in g_k(z-x)+T(z)\ \text{ for any } z\in X \text{ and } y\in T(z).$$

In particular,

$$\bar{x}\in P_{\bar{x}}^{k^{-1}}(\bar{y})\ \text{ for each } (\bar{x},\bar{y})\in\text{gph}T.$$

For each $x\in X$ and $y\in T(x)$, we define the mapping $Z_x^k: X\rightrightarrows Y$ by

$$Z_x^k(\cdot):=y+g_k(\cdot-\bar{x})-g_k(\cdot-x),$$

and the set-valued mapping $\Phi_x^k\colon X\rightrightarrows 2^Y$ by

$$\Phi_x^k(\cdot):=P_{\bar{x}}^{k^{-1}}[Z_x^k(\cdot)]. \tag{3.9}$$

Then

$$\begin{aligned}\|Z_x^k(x')-Z_x^k(x'')\| &= \|y+g_k(x'-\bar{x})-g_k(x'-x)-y-g_k(x''-\bar{x})+g_k(x''-x)\|\\ &\le \|g_k(x'-\bar{x})-g_k(x''-\bar{x})\|+\\ &\quad\ \|g_k(x'-x)-g_k(x''-x)\|\ \text{ for each } x',x''\in X.\end{aligned} \tag{3.10}$$

Now, we prove the uniformity of the semi-local convergence of the sequence generated by Algorithm 2 for solving (1.1) when T is metrically regular.

Theorem 3.1. *Suppose $\eta > 1$ and that $P^k_{\bar{x}}(\cdot)$ is metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\dfrac{\kappa}{1-\kappa\lambda}$, and let $\bar{r}$ be defined in (3.4). Let $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \subseteq O$, $\delta > 0$ and $\sigma > 0$ be such that*

(a) $\delta \le \min\left\{\dfrac{r_{\bar{x}}}{2}, \dfrac{\bar{r}}{3\lambda}, \dfrac{r_{\bar{y}}}{3\lambda}, 1\right\}$,

(b) $(\eta + 3)\kappa\lambda \le 1$,

(c) $\sigma < \lambda\delta$.

Then, for every $y \in \mathbb{B}_\sigma(\bar{y})$, there exists some $\hat{\delta} > 0$ such that any sequence $\{x_k\}$ generated by Algorithm 2 with initial point in $\mathbb{B}_{\hat{\delta}}(\bar{x})$ converges to a solution x of (1.1) for y.

Proof. Let

$$M := \frac{\kappa}{1 - 3\lambda\kappa}.$$

Since $\eta > 1$, by assumption (b) we have that

$$M\eta\lambda \le \frac{\dfrac{\eta}{\eta+3}}{1 - \dfrac{3}{\eta+3}} = 1.$$

Assumption (c) and (3.3) allow us to take $0 < \hat{\delta} \le \delta$ so that

$$\operatorname{dist}(y, T(x_0)) \le \sigma < \lambda\delta \quad \text{for each } x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x}). \tag{3.11}$$

Note that the metric regularity of the mapping $P^k_{\bar{x}}(\cdot)$ at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\dfrac{\kappa}{1-\lambda\kappa}$, implies through lemma 2.1 that ${P^k_{\bar{x}}}^{-1}$ is Lipschitz-like at $(\bar{y}, \bar{x})$ on $\mathbb{B}_{r_{\bar{y}}}(\bar{y}) \times \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with constant $\dfrac{\kappa}{1-\lambda\kappa}$, that is, (3.7) holds.

To complete the proof we will proceed by mathematical induction. It suffices to show that the Algorithm 2 generates at least one sequence and any generated sequence $\{x_k\}$ satisfies

$$\|x_k - \bar{x}\| \le 2\delta, \tag{3.12}$$

and

$$\|x_{k+1} - x_k\| \le (M\eta\lambda)^{k+1}\delta \tag{3.13}$$

for each $k = 0, 1, 2, \ldots$. Define

$$\hat{r}_x := \frac{5\kappa}{3(1-\lambda\kappa)}\big(\lambda\|x - \bar{x}\| + \|y - \bar{y}\|\big) \quad \text{for each } x \in X. \tag{3.14}$$

Since $\eta > 1$, by assumption (b) and (c) we have

$$\hat{r}_x \le \frac{5\kappa\lambda}{1-\lambda\kappa}\delta \le \frac{5}{\eta+2}\delta \le 2\delta \;\; \text{for each } x \in \mathbb{B}_{2\delta}(\bar{x}). \tag{3.15}$$

First, we will prove that

$$\mathcal{D}^0(x_0) \cap \hat{r}_{\bar{x}}(0) \neq \emptyset. \tag{3.16}$$

To do this, we will consider the mapping $\Phi^0_{x_0}$ defined by (3.9) and apply Lemma 2.3 to $\Phi^0_{x_0}$ with $\eta_0 := \bar{x}$, $r := \hat{r}_{x_0}$ and $\alpha := \dfrac{2}{5}$. It's sufficient to show that assertions (2.1) and (2.2) of Lemma 2.3

hold for $\Phi^0_{x_0}$ with $\eta_0 := \bar{x}$, $r := \hat{r}_{x_0}$ and $\alpha := \frac{2}{5}$.

To proceed, we note that $\bar{x} \in P^{0^{-1}}_{\bar{x}}(\bar{y}) \cap \mathbb{B}_{\hat{r}_{x_0}}(\bar{x})$. Then by the definition of $\Phi^0_{x_0}$ and excess e, we have

$$\begin{aligned} \operatorname{dist}(\bar{x}, \Phi^0_{x_0}(\bar{x})) &\leq e(P^{0^{-1}}_{\bar{x}}(\bar{y}) \cap \mathbb{B}_{\hat{r}_{x_0}}(\bar{x}), \Phi^0_{x_0}(\bar{x})) \\ &\leq e(P^{0^{-1}}_{\bar{x}}(\bar{y}) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), P^{0^{-1}}_{\bar{x}}[Z^0_{x_0}(\bar{x})]). \end{aligned} \tag{3.17}$$

(noting that $\mathbb{B}_{\hat{r}_{x_0}}(\bar{x}) \subseteq \mathbb{B}_{2\delta}(\bar{x}) \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$). For each $x \in \mathbb{B}_{2\delta}(\bar{x})$, we have that

$$\begin{aligned} \|Z^0_{x_0}(x) - \bar{y}\| &= \|y + g_0(x - \bar{x}) - g_0(x - x_0) - \bar{y}\| \\ &\leq \|y - \bar{y}\| + \|g_0(x - \bar{x}) - g_0(x - x_0)\| \\ &\leq \|y - \bar{y}\| + \lambda_0\|x_0 - \bar{x}\| \leq \|y - \bar{y}\| + \lambda\|x_0 - \bar{x}\|. \end{aligned} \tag{3.18}$$

Then by the relations $\sigma < \lambda\delta$ and $3\lambda\delta \leq r_{\bar{y}}$ in assumptions (c) and (a) respectively, we obtain that

$$\|Z^0_{x_0}(x) - \bar{y}\| \leq 3\lambda\delta \leq r_{\bar{y}}, \tag{3.19}$$

that is, for each $x \in \mathbb{B}_{2\delta}(\bar{x})$, $Z^0_{x_0}(x) \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. Put $x = \bar{x}$ in (3.18), we obtain that

$$\begin{aligned} \|Z^0_{x_0}(\bar{x}) - \bar{y}\| &= \|y + g_0(\bar{x} - \bar{x}) - g_0(\bar{x} - x_0) - \bar{y}\| \\ &\leq \|y - \bar{y}\| + \|g_0(0) - g_0(\bar{x} - x_0)\| \\ &\leq \|y - \bar{y}\| + \lambda_0\|x_0 - \bar{x}\| \leq \|y - \bar{y}\| + \lambda\|x_0 - \bar{x}\| \\ &\leq 3\lambda\delta \leq r_{\bar{y}}. \end{aligned} \tag{3.20}$$

This yields that $Z^0_{x_0}(\bar{x}) \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. Using (3.7), (3.20) and (3.14) in (3.17), we have

$$\operatorname{dist}(\bar{x}, \Phi^0_{x_0}(\bar{x})) \leq \frac{\kappa}{1-\lambda\kappa}\|\bar{y} - Z^0_{x_0}(\bar{x})\| \leq \frac{\kappa}{1-\lambda\kappa}(\lambda\|x_0 - \bar{x}\| + \|y - \bar{y}\|) = \left(1 - \frac{2}{5}\right)\hat{r}_{x_0} = (1-\alpha)r.$$

This implies that assertion (2.1) of Lemma 2.3 is satisfied.

Below, we will show that the assertion (2.2) of Lemma 2.3 also holds. To show this, let $x', x'' \in \mathbb{B}_{\hat{r}_{x_0}}(\bar{x})$. Then by the fact $2\delta \leq r_{\bar{x}}$ in assumption (a) and (3.15), we have $x', x'' \in \mathbb{B}_{\hat{r}_{x_0}}(\bar{x}) \subseteq \mathbb{B}_{2\delta}(\bar{x}) \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$. Moreover, we have from (3.19) that $Z^0_{x_0}(x'), Z^0_{x_0}(x'') \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. Then by Lipschitz-like property of $P^{0^{-1}}_{\bar{x}}(\cdot)$, we have

$$\begin{aligned} e(\Phi^0_{x_0}(x') &\cap \mathbb{B}_{\hat{r}_{x_0}}(\bar{x}), \Phi^0_{x_0}(x'')) \\ &\leq e(\Phi^0_{x_0}(x') \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), \Phi^0_{x_0}(x'')) \\ &= e(P^{0^{-1}}_{\bar{x}}[Z^0_{x_0}(x')] \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), P^{0^{-1}}_{\bar{x}}[Z^0_{x_0}(x'')]) \\ &\leq \frac{\kappa}{1-\lambda\kappa}\|Z^0_{x_0}(x') - Z^0_{x_0}(x'')\|. \end{aligned} \tag{3.21}$$

Applying (3.10) and (3.6) in (3.21), we obtain

$$\begin{aligned} e(\Phi^0_{x_0}(x') &\cap \mathbb{B}_{\hat{r}_{x_0}}(\bar{x}), \Phi^0_{x_0}(x'')) \\ &\leq \frac{\kappa}{1-\lambda\kappa}(\|g_0(x' - \bar{x}) - g_0(x'' - \bar{x})\| + \|g_0(x' - x_0) - g_0(x'' - x_0)\|) \\ &\leq \frac{2\lambda_0\kappa}{1-\lambda\kappa}\|x' - x''\| \leq \frac{2\lambda\kappa}{1-\lambda\kappa}\|x' - x''\| \\ &\leq \frac{2}{5}\|x' - x''\| = \alpha\|x' - x''\|. \end{aligned}$$

Therefore, the assertion (2.2) of Lemma 2.3 is also satisfied. Since both assertions (2.1) and (2.2) of Lemma 2.3 are fulfilled, there exists a fixed point

$$\hat{x}_1 \in \mathbb{B}_{\hat{r}_{x_0}}(\bar{x}) \text{ such that } \hat{x}_1 \in \Phi^0_{x_0}(\hat{x}_1),$$

which translates to $Z^0_{x_0}(\hat{x}_1) \in P^0_{\bar{x}}(\hat{x}_1)$, that is, $y \in g_0(\hat{x}_1 - x_0) + T(\hat{x}_1)$. This shows that $\hat{x}_1 - x_0 \in \mathcal{D}^0(x_0)$ and hence (3.16) holds. Consequently, inasmuch as $\eta > 1$, we can choose $d_0 \in \mathcal{D}^0(x_0)$ such that

$$\|d_0\| \leq \eta \ \mathrm{dist}(0, \mathcal{D}^0(x_0)). \tag{3.22}$$

By Algorithm 2, $x_1 := x_0 + d_0$ is defined. Hence the point x_1 is generated by Algorithm 2. Furthermore, by the definition of $\mathcal{D}^0(x_0)$, from (3.8) we can write

$$\mathcal{D}^0(x_0) := \{d_0 \in X : x_0 + d_0 \in P^{0^{-1}}_{x_0}(y)\},$$

and so

$$\mathrm{dist}(0, \mathcal{D}^0(x_0)) = \mathrm{dist}(x_0, P^{0^{-1}}_{x_0}(y)). \tag{3.23}$$

Since $P^k_{\bar{x}}(\cdot)$ is metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \times \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\dfrac{\kappa}{1 - \lambda\kappa}$, it follows from Lemma 3.1 that the mapping $P^{k^{-1}}_x(\cdot)$ is Lipschitz-like at $(\bar{y}, \bar{x})$ on $\mathbb{B}_{\bar{r}}(\bar{y}) \times \mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ with constant M for each $x \in \mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$. In particular, $P^{0^{-1}}_{x_0}(\cdot)$ is Lipschitz-like at $(\bar{y}, \bar{x})$ on $\mathbb{B}_{\bar{r}}(\bar{y}) \times \mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ with constant M as the ball $\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ contains the point $\bar{x}$. Furthermore, the facts $3\lambda\delta \leq \bar{r}$ and $\sigma < \lambda\delta$ in assumptions (a) and (c) respectively imply that

$$\sigma < \lambda\delta \leq \frac{\bar{r}}{3} < \bar{r},$$

and hence we have that $y \in \mathbb{B}_\sigma(\bar{y}) \subseteq \mathbb{B}_{\bar{r}}(\bar{y})$. Applying Lemma 2.1 we have that the mapping $P^0_{x_0}(\cdot)$ is metrically regular at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x}) \times \mathbb{B}_{\bar{r}}(\bar{y})$ with constant M, that is,

$$\mathrm{dist}(x_0, P^{0^{-1}}_{x_0}(y)) \leq M \ \mathrm{dist}(y, P^0_{x_0}(x_0)) \text{ for each } x_0 \in \mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x}) \text{ and } y \in \mathbb{B}_{\bar{r}}(\bar{y}). \tag{3.24}$$

Using (3.23), (3.24) and ((3.11)) in (3.22), we obtain that

$$\begin{aligned} \|x_1 - x_0\| = \|d_0\| &\leq \eta \ \mathrm{dist}(0, \mathcal{D}^0(x_0)) = \eta \ \mathrm{dist}(x_0, P^{0^{-1}}_{x_0}(y)) \\ &\leq \eta M \ \mathrm{dist}(y, P^0_{x_0}(x_0)) = \eta M \ \mathrm{dist}(y, T(x_0)) \\ &\leq (M\eta\lambda)\delta. \end{aligned}$$

This shows that (3.13) holds for $k = 0$.

We assume that the points $x_1, \ldots, x_n$ are generated by Algorithm 2 such that (3.12) and (3.13) are true for $k = 0, 1, 2, \ldots, n-1$. We show that there exists x_{n+1} such that (3.12) and (3.13) hold for $k = n$. Because of (3.12) and (3.13) hold for $k \leq n-1$, we have the following inequality

$$\|x_n - \bar{x}\| \leq \sum_{i=0}^{n-1} \|d_i\| + \|x_0 - \bar{x}\| \leq \delta \sum_{i=0}^{n-1} (M\eta\lambda)^{i+1} + \delta \leq \frac{M\eta\lambda}{1 - M\eta\lambda}\delta + \delta \leq 2\delta,$$

and so $x_n \in \mathbb{B}_{2\delta}(\bar{x})$. This reflects that (3.12) holds for $k = n$. Now with almost the same argument as we used for the case when $k = 0$, we can find that the mapping $P^{n^{-1}}_{x_n}(\cdot)$ is also Lipschitz-like at

$(\bar{y},\bar{x})$ on $\mathbb{B}_{\bar{r}}(\bar{y})\times\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ with constant M. Then by applying again Algorithm 2, we have

$$\begin{aligned}\|x_{n+1}-x_n\|=\|d_n\| &\leq \eta\ \mathrm{dist}(0,D^n(x_n))=\eta\ \mathrm{dist}(x_n,P_{x_n}^{n\ -1}(y))\\ &\leq \eta M\ \mathrm{dist}(y,P_{x_n}^n(x_n))=\eta M\ \mathrm{dist}(y,T(x_n))\\ &= \eta M\ \mathrm{dist}(y,y-g_{n-1}(x_n-x_{n-1}))\\ &= \eta M\ \|g_{n-1}(x_n-x_{n-1})-g_{n-1}(0)\|\\ &\leq \eta\lambda_{n-1}M\ \|x_n-x_{n-1}\|\leq\eta\lambda M\ \|x_n-x_{n-1}\| \qquad (3.25)\\ &\leq M\eta\lambda(M\eta\lambda)^n\delta\leq(M\eta\lambda)^{n+1}\delta.\end{aligned}$$

This shows that (3.13) holds for $k=n$. Hence (3.12) and (3.13) hold for each k. This implies that $\{x_n\}$ is a Cauchy sequence which is generated by Algorithm 2 and there exists $x^*\in\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ such that $x_n\to x^*$. Thus, passing to the limit $x_{n+1}\in P_{x_n}^{n^{-1}}(y)$ and since $\mathrm{gph}T\cap(\mathbb{B}_{r_{\bar{x}}}(\bar{x})\times\mathbb{B}_{r_{\bar{y}}}(\bar{y}))$ is closed, it follows that $y\in T(x^*)$. This completes the proof. □

In the particular case, when $\bar{x}$ is a solution of (1.1) for $y=0$, Theorem 3.1 can be reduced to the following corollary which gives the uniformity of the local convergence result for GG-PPA defined by Algorithm 2.

Corollary 3.1. *Suppose that $\eta>1$ and $\bar{x}$ is a solution of (1.1) for $y=0$. Let T be metrically regular at $(\bar{x},0)$ which have locally closed graph at $(\bar{x},0)$. Let $\tilde{r}>0$ be such that $\mathbb{B}_{2\tilde{r}}(\bar{x})\subseteq O$ and suppose that*

$$\lim_{x\to\bar{x}}\mathrm{dist}(0,T(x))=0. \qquad (3.26)$$

Then there exist constants $\hat{\delta}>0$ and $\sigma>0$ such that for every $y\in\mathbb{B}_\sigma(0)$ there exists any sequence $\{x_k\}$ generated by Algorithm 2 with initial point $x_0\in\mathbb{B}_{\hat{\delta}}(\bar{x})$, which is convergent to a solution x of (1.1) for y.

Proof. Since $\mathrm{gph}T$ is locally closed at $(\bar{x},0)$ and T is metrically regular at $(\bar{x},0)$, there exist constants $r_{\bar{x}},r_0>0$ such that T is metrically regular at $(\bar{x},0)$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})\times\mathbb{B}_{r_0}(0)$ with constant κ and $\mathrm{gph}T\cap(\mathbb{B}_{r_{\bar{x}}}(\bar{x})\times\mathbb{B}_{r_0}(0))$ is closed. Since $g(\cdot-\bar{x})$ is Lipschitz continuous on $O+\bar{x}$, applying Lemma 2.2 we assume that the mapping $P_{\bar{x}}^k(\cdot)$ is metrically regular at $(\bar{x},0)$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})\times\mathbb{B}_{r_0}(0)$ with constant $\dfrac{\kappa}{1-\lambda\kappa}$.

Let $\eta>1$ and $\sup_k\lambda_k:=\lambda\in(0,1)$ be such that $\kappa\lambda\leq\dfrac{1}{\eta+3}$. Choose $r_{\bar{x}}\in(0,\tilde{r})$ and $r_{\bar{y}}\in(0,r_0)$ such that $\dfrac{r_{\bar{x}}}{2}\leq\tilde{r}$ and $r_{\bar{y}}-\dfrac{r_{\bar{x}}\lambda}{2}>0$. Then put

$$\bar{r}=\min\Big\{\frac{2r_{\bar{y}}-r_{\bar{x}}\lambda}{2},\frac{r_{\bar{x}}(1-3\kappa\lambda)}{4\kappa}\Big\}>0.$$

It follows that

$$\lambda<\min\Big\{\frac{2r_{\bar{y}}}{r_{\bar{x}}},\frac{1}{3\kappa}\Big\}.$$

Let $\delta>0$ be such that

$$\delta\leq\min\Big\{\frac{r_{\bar{x}}}{2},\frac{\bar{r}}{3\lambda},\frac{r_{\bar{y}}}{3\lambda},1\Big\}.$$

Let $y\in\mathbb{B}_\sigma(0)$. Since (3.26) holds, we can take $0<\hat{\delta}\leq\delta$ so that for each $x_0\in\mathbb{B}_{\hat{\delta}}(\bar{x})$ there exists $\bar{y}$ near 0 such that $\bar{y}\in T(\bar{x})$, that is, $\bar{y}\in P_{\bar{x}}^k(\cdot)$. Then for such $\bar{y}$ we have that $y\in\mathbb{B}_\sigma(\bar{y})$ so that

$$\|y-\bar{y}\|\leq\sigma<\lambda\delta.$$

It follows that $\mathbb{B}_{r_{\bar{y}}}(\bar{y})\subseteq\mathbb{B}_{r_0}(0)$ and hence $\mathrm{gph}T\cap(\mathbb{B}_{r_{\bar{x}}}(\bar{x})\times\mathbb{B}_{r_{\bar{y}}}(\bar{y}))$ is closed. Thus, by the property of $P_{\bar{x}}^k(\cdot)$, we conclude that $P_{\bar{x}}^k(\cdot)$ is metrically regular at $(\bar{x},\bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})\times\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\dfrac{\kappa}{1-\lambda\kappa}$. Now, it is routine to justify that all assumptions in Theorem 3.1 hold. Thus, Theorem 3.1 is applicable to complete the proof of the Corollary 3.1. □

4 Numerical Experiment

In this section, we will provide a numerical example to validate the uniformity of semi-local convergence result of the GG-PPA.

Example 4.1. *Let* $X = Y = \mathbb{R}, x_0 = -0.2, y = 0.1, \eta = 1.5, \lambda = 0.3$ and $\kappa = 0.2$. *Define a set-valued mapping* T *on* $\mathbb{R}$ *by* $T(x) = \{3x+1, 4\}$. *Consider a sequence of Lipschitz continuous function* g_n *with* $g_n(0) = 0$, *which is defined by* $g_n(x) = -\frac{1}{2}x$. *Then Algorithm 2 generates a sequence which converges to* $x^* = -0.3$ *for* 0.1.

From the statement, it is obvious that T is metrically regular at $(-0.2, 0.4) \in \text{gph}\, T$ and g_n is Lipschitz continuous in the neighborhood of origin with Lipschitz constant $\sup_k \lambda_k := \lambda = 0.3$. Consider $T(x) := 3x + 1$. Thus from (1.3), we have that

$$\begin{aligned} D^k(x_k) &= \Big\{d_k \in \mathbb{R} : y \in g_k(d_k) + T(x_k + d_k)\Big\} \\ &= \Big\{d_k \in \mathbb{R} : d_k = \frac{-(30x_k + 9)}{25}\Big\}. \end{aligned}$$

On the other hand, if $D^k(x_k) \neq \emptyset$ we obtain that

$$y \in g_k(x_{k+1} - x_k) + T(x_{k+1}) \Rightarrow x_{k+1} = \frac{-(5x_k + 9)}{25}.$$

Thus from (3.25), we obtain that

$$\|d_k\| \leq \frac{\eta\kappa\lambda}{1 - 3\kappa\lambda}\|d_{k-1}\|.$$

For the given values of η, λ, κ, we see that $\frac{\eta\kappa\lambda}{1 - 3\kappa\lambda} = \frac{9}{82} < 1$. Thus, this implies that the sequence generated by Algorithm 2 converges linearly. Then the following Table 1, obtained by using Mat lab program, indicates that the solution of the assumed generalized equation is -0.3 for 0.1 when $k = 10$.

Table 1. Finding a solution of generalized equation

x	T(x)
-0.2000	0.4000
-0.3200	0.0400
-0.2960	0.1120
-0.3008	0.0976
-0.2998	0.1005
-0.3000	0.0999
-0.3000	0.1000
-0.3000	0.1000
-0.3000	0.1000
-0.3000	0.1000
-0.3000	0.1000

The following figure is the graphical representation of $T(x)$

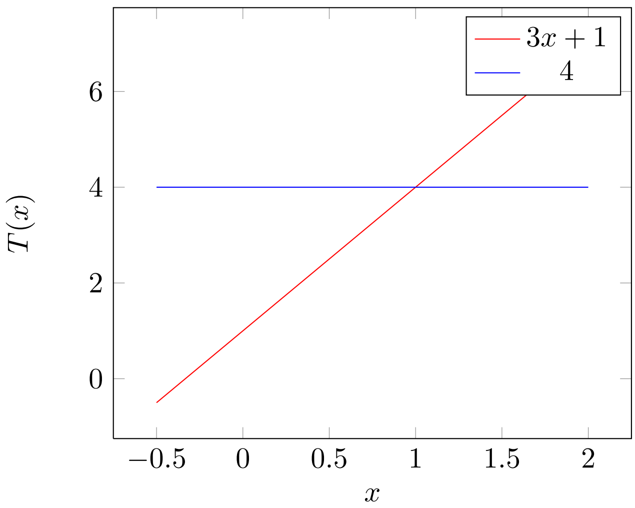

Fig. 1. The graph of $T(x)$

5 Concluding Remarks

Semi-local and local convergence results for the GG-PPA, defined by Algorithm 2, are presented under yellow the assumptions that $\eta > 1$, T is metrically regular at a given point which have locally closed graph and a sequence of Lipschitz continuous functions g_k with $g_k(0) = 0$. We observe that in the case when $g_k(u) = \lambda_k u$, Algorithm 2 is reduced to the classical Gauss-type proximal point algorithm introduced by Rashid in his PhD thesis [4, Algorithm 4.2.1]. This result improves and extends corresponding one [1][4][9][13].

Acknowledgement

We thank the editor and the referees for their valuable comments which improve this manuscript. This work is fully supported by Ministry of Science and Technology, Bangladesh, grant No. 39.009.002.01.00.053.2014-2015/246(EAS-19). We greatly appreciate this support.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Alom MA, Rashid MH, Dey KK. Convergence analysis of the general version of gauss-type proximal point method for metrically regular mappings [J]. J. Applied Mathematics. 2016;7(11):1248-1259.

[2] Robinson SM. Generalized equations and their solutions, part I: basic theory [J]. Math. Program. Stud. 1979;10:128-141.

[3] Robinson SM. Generalized equations and their solutions, part II: applications to nonlinear programing [J]. Math. Program. Stud. 1982;19:200-221.

[4] Rashid MH. Iteration methods for solving generalized equations in Banach spaces. PhD thesis, Zhejiang University; 2012.

[5] Rashid MH. Convergence analysis of gauss-type proximal point method for variational inequalities [J]. Open Science Journal of Mathematics and Application. 2014;2(1):5-14.

[6] Martinet B. Régularisation d'inéquations variationnelles par approximations successives [J]. Rev. Fr. Inform. Rech. Opér. 1970;3:154-158.

[7] Mordukhovich BS. Variational Analysis and generalized differentiation I: Basic theory [B]. Grundlehren Math. Wiss., Springer-Verlag, Berlin; 2006;330.

[8] Pennanen T. Local convergence of the proximal point algorithm and multiplier methods without monotonicity [J]. Math. Oper. Res. 2002;27:170-191.

[9] Rashid MH, Wang JH, Li C. Convergence analysis of gauss-type proximal point method for metrically regular mappings [J]. J. Nonlinear and Convex Analysis. 2013;14(3):627-635.

[10] Rockafellar RT, Wets RJB. Variational analysis [B]. Springer-Verlag, Berlin; 1997.

[11] Rockafellar RT. Monotone operators and the proximal point algorithm [J]. SIAM J. Control Optim. 1976;14:877-898.

[12] Aragón Artacho FJ, Dontchev AL, Geoffroy MH. Convergence of the proximal point method for metrically regular mappings [J]. ESAIM: Proceedings. 2007;17:1-8.

[13] Aragón Artacho FJ, Geoffroy MH. Uniformity and inexact version of a proximal point method for metrically regular mappings [J]. J. Math. Anal. Appl. 2007;335:168-183.

[14] Li C, Zhang WH, Jin XQ. Convergence and uniqueness properties of Gauss-Newton's method [J].Comput. Math. Appl. 2004;47:1057-1067.

[15] Rashid MH, Yu SH, Li C, Wu SY. Convergence analysis of the Gauss-Newton-type method for Lipschitz-like mappings [J]. J. Optim. Theory Appl. 2013;158(1):216-233.

[16] Rashid MH. On the convergence of extended Newton-type method for solving variational inclusions [J]. Journal of Cogent Mathematics. 2014;1(1):1-19.

[17] Dontchev AL, Rockafellar RT. Implicit functinos and solution mappings: A view from variational analysis [B]. Springer Science+Business Media, LLC, New York; 2009.

[18] Dontchev AL, Rockafellar RT. Regularity and conditioning of solution mappings in variational analysis [J]. Set-valued Anal. 2004;12(1):79-109.

[19] Mordukhovich BS. Complete characterization of opennes, metric regularity, and Lipschitzian properties of multifunctions [J]. Trans. Amer. Math. Soc. 1993;340(1):1-35.

[20] Aubin JP, Frankowska H. Set-valued analysis [B]. Birkhäuser, Boston; 1990.

[21] Dontchev AL, Lewis AS, Rockafellar RT. The radius of metric regularity [J]. Trans. AMS. 2002;355:493-517.

[22] Dontchev AL, Hager WW. An inverse mapping theorem for set-valued maps [J]. Proc. Amer. Math. Soc. 1994;121:481-498.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/17880