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Coverage Probability of the Credible Interval and Credible Probability of the Confidence Interval of the Hierarchical Normal Model

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Authors' contributions

This work was carried out in collaboration between both authors. Author YYZ proved the two theorems, and wrote the first draft of the manuscript. Author TZR did literature searches and revised the manuscript. Both authors read and approved the final manuscript.

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Abstract

It is well known that the coverage probability of a given nominal level confidence interval and the credible probability of a given nominal level credible interval will attain the nominal level. Moreover, it is commonly believed that the two switching concepts probabilities, that is, the coverage probability of a given nominal level credible interval and the credible probability of a given nominal level confidence interval, can not attain the nominal level in general. For the hierarchical normal model, we show that the two switching concepts probabilities can attain the nominal level in the limit when a skillful classified variable is infinity. The numerical simulations illustrate the correctness of our findings.

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1 Introduction

Statistical inferences are covered in many classical textbooks [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. The main estimates of statistical inferences are point estimates and interval estimates. The usual interval estimates are confidence intervals and credible intervals. The confidence intervals are usually measured under the coverage probability, while the credible intervals are usually measured under the credible probability. There is no article research on the credible probability of the confidence interval. However, there are a lot of research on the coverage probability of the credible interval. [11] find that allowing for genotyping error yielded relative risk estimates that were approximately unbiased, together with 95% credible intervals giving approximately correct coverage probability. [12] find that the Bayesian credible intervals based on the same priors also have super frequentist coverage probabilities while some of the frequentist confidence intervals procedures have drastically poor coverage. [13] find that the impact of different choices of prior distributions on the coverage probability of credible intervals is unknown. [14] compare three Bayesian sample size criteria: the Average Coverage Criterion (ACC) which controls the coverage rate of fixed length credible intervals over the predictive distribution of the data, the Average Length Criterion (ALC) which controls the length of credible intervals with a fixed coverage rate, and the Worst Outcome Criterion (WOC) which ensures the desired coverage rate and interval length over all (or a subset of) possible datasets. [15] find that Bayesian interval estimates for the treatment effect are longer on average, though there is little improvement in coverage probability. [16] find that the 95% confidence/credible intervals also possess good coverage properties, given that the point estimates perform good. [17] find that the coverage probability of the credible interval close to the nominal value, with a small coverage asymmetry in some cases. [18] find that 95% credible intervals may not retain nominal coverage, and treatment rank probabilities may become distorted. [19] find that Bayesian inference provides reliable credible intervals in terms of bias and coverage probability. [20] carry out simulations and find that spatial capture-recapture models produced more accurate parameter estimates with better credible interval coverage than non-spatial capture-recapture models. [21] find that the coverage probability of given credible interval is well-calibrated in the simulation experiments.

It is well known that the coverage probability of a given nominal level $1-\alpha$ confidence interval and the credible probability of a given nominal level $1-\alpha$ credible interval will attain the nominal level $1-\alpha$. In Example 9.2.18, [4] tell us that for the hierarchical normal model, the two switching concepts probabilities, that is, the coverage probability of the $1-\alpha$ credible interval and the credible probability of the $1-\alpha$ confidence interval, can not attain the nominal level $1-\alpha$ in general. Whether it is possible, under some circumstances, that the two switching concepts probabilities attain the nominal level $1-\alpha$ is not mentioned in the example and the relevant exercises in [4]. Nobody has considered this problem to the best of our knowledge. By inspecting the calculations in Example 9.2.18, we find that they use a specific configuration for $\tau(n) = \sigma/\sqrt{n}$ which is a known function of n. Inspired by this specific configuration, we show that the two switching concepts probabilities can attain the nominal level $1-\alpha$ in the limit by taking suitable configurations for $\tau(n)$. The numerical simulations illustrate the correctness of our findings.

The rest of the paper is organized as follows. In the next Section 2, we prove two theorems for the hierarchical normal model. Theorem 2.1 concerns the limiting behavior of the coverage probability of the $1-\alpha$ credible interval. Theorem 2.2 concerns the limiting behavior of the credible probability of the $1-\alpha$ confidence interval. In Section 3, the numerical simulations are carried out to illustrate the correctness of the two theorems. Section 4 concludes.

2 Main Results

The hierarchical normal model of Example 9.2.18 in [4] is as follows. Let $X_1, \ldots, X_n$ be iid $N\left(\theta, \sigma^2\right)$, and let θ have the prior pdf $N\left(\mu, \tau^2\right)$, where μ, σ, and τ are all known. For simplicity, we assume $-\infty < \mu < \infty$ and $\sigma > 0$ are known constants. Furthermore, we assume that $\tau(n)$ is a known function of n, which is inspired by Example 9.2.18 in [4]. There, they use $\tau(n) = \sigma/\sqrt{n}$ which is a known function of n. In fact, from theorems 1 and 2 below, we find that $\tau(n)$ as a function of n is critical to define the skillful classified variable $L = \lim_{n\to\infty} \sqrt{n}\tau^2(n)$.

For the coverage probability of the $1-\alpha$ credible interval of the hierarchical normal model $P_n(\theta)$, from Example 9.2.18 and by noting that $\gamma = \frac{\sigma^2}{n\tau^2(n)}$, we have

$$\begin{aligned} P_n(\theta) &= P_\theta\left(\left|Z - \frac{\gamma(\theta-\mu)}{\sigma/\sqrt{n}}\right| \le z_{\frac{\alpha}{2}}\sqrt{1+\gamma}\right) \\ &= P_\theta\left(\left|Z - \frac{\sigma(\theta-\mu)}{\sqrt{n}\tau^2(n)}\right| \le z_{\frac{\alpha}{2}}\sqrt{1+\frac{\sigma^2}{n\tau^2(n)}}\right), \end{aligned}$$

where $Z \sim N(0,1)$ and $z_{\frac{\alpha}{2}}$ is the $\frac{\alpha}{2}$ upper quantile of Z. Let

$$A(n) = \frac{\sigma(\theta-\mu)}{\sqrt{n}\tau^2(n)} \text{ and } B(n) = \sqrt{1+\frac{\sigma^2}{n\tau^2(n)}}.$$

Then

$$\begin{aligned} P_n(\theta) &= P_\theta\left(|Z - A(n)| \le z_{\frac{\alpha}{2}} B(n)\right) \\ &= P_\theta\left(A(n) - z_{\frac{\alpha}{2}} B(n) \le Z \le A(n) + z_{\frac{\alpha}{2}} B(n)\right). \end{aligned} \tag{2.1}$$

We have the following theorem which concerns the limiting behavior of the coverage probability of the $1-\alpha$ credible interval. Note that the choice of $L = \lim_{n\to\infty} \sqrt{n}\tau^2(n)$ as the classified variable is skillful.

Theorem 2.1. *Let $L = \lim_{n\to\infty} \sqrt{n}\tau^2(n)$ and assume $\theta \ne \mu$. Then*

$$\lim_{n\to\infty} P_n(\theta) = \begin{cases} 1-\alpha, & \text{if } L = \infty, \\ 0, & \text{if } L = 0, \\ P_\theta\left(\left|Z - \frac{\sigma(\theta-\mu)}{L}\right| \le z_{\frac{\alpha}{2}}\right), & \text{if } L \in (0,\infty). \end{cases}$$

Proof. If $L = \infty$, then

$$\begin{aligned} A(n) &= \frac{\sigma(\theta-\mu)}{\sqrt{n}\tau^2(n)} \to \frac{\sigma(\theta-\mu)}{L} = 0, \text{ as } n \to \infty, \\ B(n) &= \sqrt{1+\frac{\sigma^2}{n\tau^2(n)}} \to \sqrt{1+\frac{\sigma^2}{\infty}} = 1, \text{ as } n \to \infty. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} P_n(\theta) = P_\theta\left(|Z| \le z_{\frac{\alpha}{2}}\right) = 1-\alpha.$$

If $L = 0$, that is, $\sqrt{n}\tau^2(n) \to 0$, as $n \to \infty$. Then

$$\tau(n),\ \tau^2(n),\ n^{\frac{1}{4}}\tau(n) \to 0, \text{ as } n \to \infty. \tag{2.2}$$

When $\theta > \mu$, we have $A(n) \to \infty$, as $n \to \infty$. We want to show that the lower bound in (2.1)

$$A(n) - z_{\frac{\alpha}{2}} B(n) \to \infty, \text{ as } n \to \infty.$$

But $B(n)$ may also tend to ∞, as $n \to \infty$. We have

$$\begin{aligned} A(n) - z_{\frac{\alpha}{2}} B(n) &= \frac{\sigma(\theta-\mu)}{\sqrt{n}\tau^2(n)} - \sqrt{1 + \frac{\sigma^2}{n\tau^2(n)}} z_{\frac{\alpha}{2}} \\ &= \frac{\sigma(\theta-\mu)}{\sqrt{n}\tau^2(n)} - \frac{\sqrt{n\tau^2(n)+\sigma^2}}{\sqrt{n}\tau(n)} z_{\frac{\alpha}{2}} \\ &= \frac{1}{\sqrt{n}\tau^2(n)} \left[\sigma(\theta-\mu) - \tau(n)\sqrt{n\tau^2(n)+\sigma^2} z_{\frac{\alpha}{2}}\right]. \end{aligned}$$

By (2.2), the limit

$$\begin{aligned} &\lim_{n\to\infty} \left[\tau(n)\sqrt{n\tau^2(n)+\sigma^2} z_{\frac{\alpha}{2}}\right] = \lim_{n\to\infty} \left[n^{\frac{1}{4}}\tau(n) \frac{\sqrt{n\tau^2(n)+\sigma^2}}{n^{\frac{1}{4}}} z_{\frac{\alpha}{2}}\right] \\ &= \lim_{n\to\infty} \left[n^{\frac{1}{4}}\tau(n) \sqrt{\sqrt{n}\tau^2(n) + \frac{\sigma^2}{\sqrt{n}}} z_{\frac{\alpha}{2}}\right] = \lim_{n\to\infty}\left[n^{\frac{1}{4}}\tau(n)\right] \sqrt{\lim_{n\to\infty}\left[\sqrt{n}\tau^2(n)\right] + \lim_{n\to\infty}\frac{\sigma^2}{\sqrt{n}}} z_{\frac{\alpha}{2}} \\ &= 0 \cdot \sqrt{0+0} z_{\frac{\alpha}{2}} = 0. \end{aligned}$$

Since $\frac{1}{\sqrt{n}\tau^2(n)} \to \infty$, as $n \to \infty$, and $\sigma(\theta-\mu) > 0$, we therefore have

$$\lim_{n\to\infty} \left[A(n) - z_{\frac{\alpha}{2}} B(n)\right] = \infty.$$

Consequently, $\lim_{n\to\infty} P_n(\theta) = 0$. When $\theta < \mu$, we have $A(n) \to -\infty$, as $n \to \infty$. Similarly, we can prove that the upper bound in (2.1)

$$A(n) + z_{\frac{\alpha}{2}} B(n) \to -\infty, \text{ as } n \to \infty.$$

Consequently, $\lim_{n\to\infty} P_n(\theta) = 0$.

If $L \in (0, \infty)$, then

$$\begin{aligned} A(n) &= \frac{\sigma(\theta-\mu)}{\sqrt{n}\tau^2(n)} \to \frac{\sigma(\theta-\mu)}{L}, \text{ as } n \to \infty, \\ B(n) &= \sqrt{1 + \frac{\sigma^2}{n\tau^2(n)}} \to \sqrt{1 + \frac{\sigma^2}{\infty}} = 1, \text{ as } n \to \infty. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} P_n(\theta) = P_\theta\left(\left|Z - \frac{\sigma(\theta-\mu)}{L}\right| \le z_{\frac{\alpha}{2}}\right) \in (0, 1-\alpha).$$

The proof of the theorem is complete. □

In Theorem 2.1, if L does not exist, then this case is complicated, and we do not pursue it. We have the following remarks for Theorem 2.1.

Remark 2.1. We can take specific configurations of the parameter $\tau^2(n)$, so that $\lim_{n\to\infty} P_n(\theta)$ can take different values in Theorem 2.1. For example, the following configurations of $\tau^2(n)$ will guarantee $L = \lim_{n\to\infty} \sqrt{n}\tau^2(n) = \infty$, and thus $\lim_{n\to\infty} P_n(\theta) = 1-\alpha$:

- $\tau^2(n) \to c \in (0,\infty)$, as $n \to \infty$. In particular, $\tau^2(n) = \tau_0^2 \to \tau_0^2 \in (0,\infty)$, as $n \to \infty$.
- $\tau^2(n) \to \infty$, as $n \to \infty$.
- $\lim_{n\to\infty} \tau^2(n)$ does not exist, and there exist an $\varepsilon > 0$ and an $N > 0$, such that for any $n \geq N$,

$$\left|\tau^2(n)\right| \geq \varepsilon > 0,$$

  that is, $\tau^2(n)$ is bounded away from 0. In particular,

$$\tau^2(n) = \begin{cases} 1, & n \text{ is odd}, \\ 2, & n \text{ is even}. \end{cases}$$

- $\tau^2(n) \to 0$, as $n \to \infty$, but $\sqrt{n}\tau^2(n) \to L = \infty$, as $n \to \infty$. For instance, $\tau^2(n) = n^{-\frac{1}{4}}$.

The configuration $\tau(n) = \sigma/\sqrt{n}$ in [4] has

$$\sqrt{n}\tau^2(n) = \sqrt{n}\frac{\sigma^2}{n} = \frac{\sigma^2}{\sqrt{n}} \to 0 = L, \text{ as } n \to \infty.$$

Thus, by Theorem 2.1, $\lim_{n\to\infty} P_n(\theta) = 0$.

If $\tau^2(n) = \frac{c}{\sqrt{n}}$, where $c \in (0,\infty)$, then

$$\sqrt{n}\tau^2(n) = \sqrt{n}\frac{c}{\sqrt{n}} = c \to c = L \in (0,\infty), \text{ as } n \to \infty.$$

Thus, by Theorem 2.1,

$$\lim_{n\to\infty} P_n(\theta) = P_\theta\left(\left|Z - \frac{\sigma(\theta-\mu)}{c}\right| \leq z_{\frac{\alpha}{2}}\right).$$

Remark 2.2. When $\theta = \mu$, it is easy to check that $A(n) = 0$. Therefore,

$$P_n(\theta) = P_\theta\left(|Z| \leq z_{\frac{\alpha}{2}}\sqrt{1+\frac{\sigma^2}{n\tau^2}}\right) = 2\Phi\left(z_{\frac{\alpha}{2}}\sqrt{1+\frac{\sigma^2}{n\tau^2}}\right) - 1,$$

where $\Phi(x)$ is the cdf of $Z \sim N(0,1)$. Note that $P_n(\theta)$ does not depend on θ. Since $\frac{\sigma^2}{n\tau^2} \in [0,\infty]$, we have $P_n(\theta) \in [1-\alpha, 1]$.

For the credible probability of the $1-\alpha$ confidence interval of the hierarchical normal model, from Example 9.2.18, we have

$$P_n(\bar{x}) = P_{\bar{x}}\left(\left|Z - \frac{\gamma(\bar{x}-\mu)}{\sqrt{1+\gamma}\sigma/\sqrt{n}}\right| \leq z_{\frac{\alpha}{2}}\sqrt{1+\gamma}\right).$$

Let

$$C(n) = \frac{\gamma(\bar{x}-\mu)}{\sqrt{1+\gamma}\sigma/\sqrt{n}} \text{ and } B(n) = \sqrt{1+\gamma}.$$

Then

$$\begin{aligned} P_n(\bar{x}) &= P_{\bar{x}}\left(|Z - C(n)| \leq z_{\frac{\alpha}{2}}B(n)\right) \\ &= P_{\bar{x}}\left(C(n) - z_{\frac{\alpha}{2}}B(n) \leq Z \leq C(n) + z_{\frac{\alpha}{2}}B(n)\right). \end{aligned} \tag{2.3}$$

Note that $\gamma = \sigma^2/\left(n\tau^2\right)$, and thus

$$B(n) = \sqrt{1+\frac{\sigma^2}{n\tau^2(n)}}$$

and

$$C(n) = \frac{\gamma(\bar{x}-\mu)}{\sqrt{1+\gamma\sigma/\sqrt{n}}} = \frac{\frac{\sigma^2}{n\tau^2}(\bar{x}-\mu)}{\sqrt{1+\frac{\sigma^2}{n\tau^2}\frac{\sigma}{\sqrt{n}}}} = \frac{\frac{\sigma^2}{n\tau^2}(\bar{x}-\mu)}{\frac{\sqrt{n\tau^2+\sigma^2}}{\sqrt{n}\tau}\frac{\sigma}{\sqrt{n}}} = \frac{\sigma^2}{n\tau^2}(\bar{x}-\mu)\frac{n\tau}{\sqrt{n\tau^2+\sigma^2}\sigma}$$
$$= \frac{\sigma(\bar{x}-\mu)}{\tau\sqrt{n\tau^2+\sigma^2}} = \frac{\sigma(\bar{x}-\mu)}{\sqrt{n\tau^4+\sigma^2\tau^2}} = \frac{\sigma(\bar{x}-\mu)}{\sqrt{\left(\sqrt{n}\tau^2\right)^2+\sigma^2\tau^2}}.$$

Analogous to Theorem 2.1, we have the following theorem which concerns the limiting behavior of the credible probability of the $1-\alpha$ confidence interval.

Theorem 2.2. *Let* $L = \lim_{n\to\infty}\sqrt{n}\tau^2(n)$ *and assume* $\bar{x}\neq\mu$. *Then*

$$\lim_{n\to\infty} P_n(\bar{x}) = \begin{cases} 1-\alpha, & \text{if } L=\infty, \\ 0, & \text{if } L=0, \\ P_{\bar{x}}\left(\left|Z-\frac{\sigma(\bar{x}-\mu)}{L}\right| \le z_{\frac{\alpha}{2}}\right), & \text{if } L\in(0,\infty). \end{cases}$$

Proof. The proof of Theorem 2.2 follows from that of Theorem 2.1. If $L=\infty$, then

$$|C(n)| = \frac{|\sigma(\bar{x}-\mu)|}{\sqrt{\left(\sqrt{n}\tau^2\right)^2+\sigma^2\tau^2}} \le \frac{|\sigma(\bar{x}-\mu)|}{\sqrt{\left(\sqrt{n}\tau^2\right)^2}} \to \frac{|\sigma(\bar{x}-\mu)|}{L} = 0, \text{ as } n\to\infty,$$
$$B(n) = \sqrt{1+\frac{\sigma^2}{n\tau^2(n)}} \to \sqrt{1+\frac{\sigma^2}{\infty}} = 1, \text{ as } n\to\infty.$$

Therefore,

$$\lim_{n\to\infty} P_n(\bar{x}) = P_{\bar{x}}\left(|Z|\le z_{\frac{\alpha}{2}}\right) = 1-\alpha.$$

If $L=0$, that is, $\sqrt{n}\tau^2(n)\to 0$, as $n\to\infty$, then (2.2) is right. When $\bar{x}>\mu$, we have $C(n)\to\infty$, as $n\to\infty$. We want to show that the lower bound in (2.3)

$$C(n) - z_{\frac{\alpha}{2}}B(n) \to \infty, \text{ as } n\to\infty.$$

But $B(n)$ may also tend to ∞, as $n\to\infty$. We have

$$C(n) - z_{\frac{\alpha}{2}}B(n) = \frac{\sigma(\bar{x}-\mu)}{\sqrt{\left(\sqrt{n}\tau^2\right)^2+\sigma^2\tau^2}} - z_{\frac{\alpha}{2}}\sqrt{1+\frac{\sigma^2}{n\tau^2}}$$
$$= \frac{1}{\sqrt{\left(\sqrt{n}\tau^2\right)^2+\sigma^2\tau^2}}\left[\sigma(\bar{x}-\mu) - z_{\frac{\alpha}{2}}\sqrt{1+\frac{\sigma^2}{n\tau^2}}\sqrt{\left(\sqrt{n}\tau^2\right)^2+\sigma^2\tau^2}\right].$$

And the limit

$$\lim_{n\to\infty}\left[\sqrt{1+\frac{\sigma^2}{n\tau^2}}\sqrt{\left(\sqrt{n}\tau^2\right)^2+\sigma^2\tau^2}\right] = \lim_{n\to\infty}\left[\sqrt{n}\tau^2\sqrt{1+\frac{\sigma^2}{n\tau^2}}\sqrt{1+\frac{\sigma^2}{n\tau^2}}\right]$$
$$= \lim_{n\to\infty}\left[\sqrt{n}\tau^2\left(1+\frac{\sigma^2}{n\tau^2}\right)\right] = \lim_{n\to\infty}\left[\sqrt{n}\tau^2+\frac{\sigma^2}{\sqrt{n}}\right] = 0+0=0.$$

Since $\sigma(\bar{x}-\mu)>0$ and

$$\frac{1}{\sqrt{\left(\sqrt{n}\tau^2\right)^2+\sigma^2\tau^2}} \to \frac{1}{\sqrt{0+0}} = \infty, \text{ as } n\to\infty,$$

we therefore have

$$\lim_{n\to\infty}\left[C\left(n\right)-z_{\frac{\alpha}{2}}B\left(n\right)\right]=\infty.$$

Consequently, $\lim_{n\to\infty} P_n\left(\bar{x}\right)=0$. When $\bar{x}<\mu$, we have $C\left(n\right)\to-\infty$, as $n\to\infty$. Similarly, we can prove that the upper bound in (2.3)

$$C\left(n\right)+z_{\frac{\alpha}{2}}B\left(n\right)\to-\infty,\text{ as } n\to\infty.$$

Consequently, $\lim_{n\to\infty} P_n\left(\bar{x}\right)=0$.

If $L\in(0,\infty)$, then

$$\begin{aligned}
C\left(n\right)&=\frac{\sigma\left(\bar{x}-\mu\right)}{\sqrt{\left(\sqrt{n}\tau^2\right)^2+\sigma^2\tau^2}}\to\frac{\sigma\left(\bar{x}-\mu\right)}{\sqrt{L^2+0}}=\frac{\sigma\left(\bar{x}-\mu\right)}{L},\text{ as } n\to\infty,\\
B\left(n\right)&=\sqrt{1+\frac{\sigma^2}{n\tau^2\left(n\right)}}\to\sqrt{1+\frac{\sigma^2}{\infty}}=1,\text{ as } n\to\infty.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}P_n\left(\bar{x}\right)=P_{\bar{x}}\left(\left|Z-\frac{\sigma\left(\bar{x}-\mu\right)}{L}\right|\le z_{\frac{\alpha}{2}}\right)\in\left(0,1-\alpha\right).$$

The proof of the theorem is complete. □

In Theorem 2.2, if L does not exist, then this case is complicated, and we do not pursue it. Note that Remark 2.1 is also suitable for Theorem 2.2. Moreover, we have the following remark for Theorem 2.2.

Remark 2.3. When $\bar{x}=\mu$, it is easy to check that $C\left(n\right)=0$. Therefore,

$$P_n\left(\bar{x}\right)=P_{\bar{x}}\left(|Z|\le z_{\frac{\alpha}{2}}\sqrt{1+\frac{\sigma^2}{n\tau^2}}\right)=2\Phi\left(z_{\frac{\alpha}{2}}\sqrt{1+\frac{\sigma^2}{n\tau^2}}\right)-1.$$

Note that $P_n\left(\bar{x}\right)$ does not depend on $\bar{x}$. Since $\frac{\sigma^2}{n\tau^2}\in[0,\infty]$, we have $P_n\left(\bar{x}\right)\in[1-\alpha,1]$.

3 Numerical Simulations

In this section, we will numerically illustrate the correctness of Theorems 2.1 and 2.2.

We first illustrate Theorem 2.1. Since we are considering the limiting behavior of the coverage probability of the $1-\alpha$ credible interval $P_n\left(\theta\right)$, θ is fixed and unknown. In our simulations, we assume that $\theta=2$. The other parameters are

$$\mu=1,\ \sigma=3,\ \alpha=0.1,$$

where μ is the prior mean, σ is the standard deviation of the normal model, and α is the significance level ($1-\alpha=0.9$ is the nominal level). We will use the configurations of the parameter $\tau\left(n\right)$ specified in Remark 2.1:

$$\tau_{11}(n) = \tau_0 = 1 \to 1 \in (0,\infty), \text{ as } n \to \infty,$$
$$\tau_{12}(n) = n \to \infty, \text{ as } n \to \infty,$$
$$\tau_{13}(n) = \begin{cases} 1, & n \text{ is odd}, \\ \sqrt{2}, & n \text{ is even}, \end{cases}$$
$$\tau_{14}(n) = n^{-\frac{1}{8}} \to 0, \text{ as } n \to \infty,$$
$$\tau_2(n) = \frac{\sigma}{\sqrt{n}},$$
$$\tau_3(n) = \frac{\sqrt{c}}{n^{\frac{1}{4}}} = \frac{\sqrt{2}}{n^{\frac{1}{4}}}.$$

It is easy to check that for the above configurations,

$$L = \lim_{n\to\infty} \sqrt{n}\tau^2(n) = \begin{cases} \infty, & \text{for } \tau(n) = \tau_{11}(n), \tau_{12}(n), \tau_{13}(n), \tau_{14}(n), \\ 0, & \text{for } \tau(n) = \tau_2(n), \\ c \in (0,\infty), & \text{for } \tau(n) = \tau_3(n). \end{cases}$$

The limiting behaviors of the coverage probabilities of the $1-\alpha$ credible intervals $P_n(\theta)$ for the six configurations of $\tau(n)$ are reported in Fig. 1. From Fig. 1, we clearly see that for the first four configurations of $\tau(n)$, $P_n(\theta) \to 1-\alpha = 0.9$, as $n \to \infty$. For $\tau(n) = \tau_2(n)$, $P_n(\theta) \to 0$, as $n \to \infty$. For $\tau(n) = \tau_3(n)$,

$$P_n(\theta) \to P_\theta\left(\left|Z - \frac{\sigma(\theta-\mu)}{c}\right| \le z_{\frac{\alpha}{2}}\right) = 0.557 \in (0, 0.9) = (0, 1-\alpha), \text{ as } n \to \infty.$$

The results of Fig. 1 illustrate the correctness of Theorem 2.1.

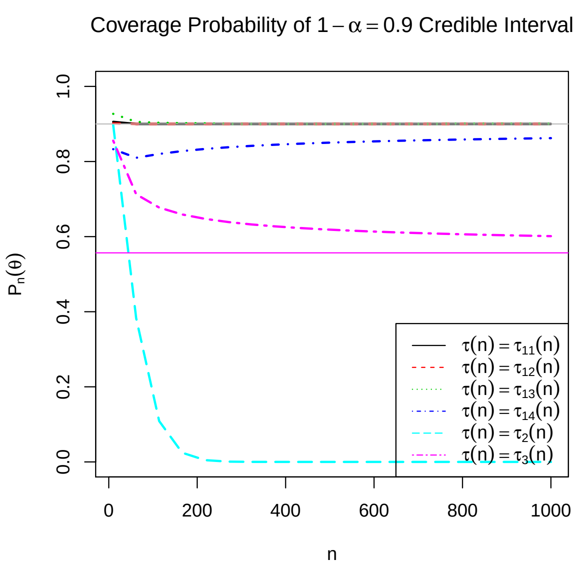

Fig. 1. The limiting behaviors of the coverage probabilities of the $1-\alpha$ credible intervals

Now we illustrate Theorem 2.2. Since we are considering the limiting behavior of the credible probability of the $1-\alpha$ confidence interval $P_n(\bar{x})$, $\bar{x}$ is fixed. In our simulations, we assume that $\bar{x}=4$. The other parameters are

$$\mu=1,\ \sigma=3,\ \alpha=0.1,$$

the same as those in the previous simulation. We will use the six configurations of the parameter $\tau(n)$ specified in the previous simulation, while c is changed to 5. For the six configurations, $L=\lim_{n\to\infty}\sqrt{n}\tau^2(n)$ is the same as those in the previous simulation. The limiting behaviors of the credible probabilities of the $1-\alpha$ confidence intervals $P_n(\bar{x})$ for the six configurations of $\tau(n)$ are reported in Fig. 2. From Fig. 2, we clearly see that for the first four configurations of $\tau(n)$, $P_n(\bar{x})\to 1-\alpha=0.9$, as $n\to\infty$. We see that for $\tau(n)=\tau_{14}(n)$, $P_n(\bar{x})$ increases very slowly. Therefore, we let n increase from 10 to 10000 (note that in Fig. 1, this number is 1000). For $\tau(n)=\tau_2(n)$, $P_n(\bar{x})\to 0$, as $n\to\infty$. For $\tau(n)=\tau_3(n)$,

$$P_n(\bar{x})\to P_{\bar{x}}\left(\left|Z-\frac{\sigma(\bar{x}-\mu)}{c}\right|\le z_{\frac{\alpha}{2}}\right)=0.438\in(0,0.9)=(0,1-\alpha)\,,\text{ as } n\to\infty.$$

The results of Fig. 2 illustrate the correctness of Theorem 2.2.

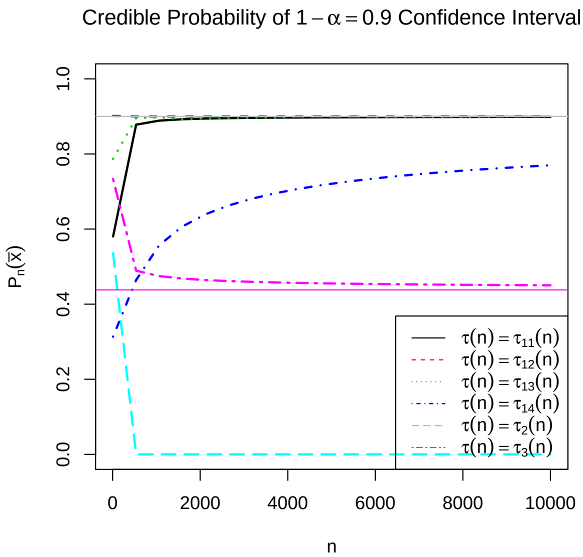

Fig. 2. The limiting behaviors of the credible probabilities of the $1-\alpha$ confidence intervals

4 Conclusion

We prove two theorems for the hierarchical normal model. Theorem 2.1 concerns the limiting behavior of the coverage probability of the $1-\alpha$ credible interval $P_n(\theta)$. Theorem 2.2 concerns the limiting behavior of the credible probability of the $1-\alpha$ confidence interval $P_n(\bar{x})$. When $L=\lim_{n\to\infty}\sqrt{n}\tau^2(n)=\infty$, the two limiting probabilities $\lim_{n\to\infty}P_n(\theta)=\lim_{n\to\infty}P_n(\bar{x})=1-\alpha$, that is, the coverage probability of the $1-\alpha$ credible interval and the credible probability of the $1-\alpha$ confidence interval attain the nominal level $1-\alpha$ in the limit. The specific configuration $\tau(n)=\sigma/\sqrt{n}$ in [4] corresponds to $L = 0$ in Theorems 2.1 and 2.2, and thus $\lim_{n\to\infty} P_n(\theta) = \lim_{n\to\infty} P_n(\bar{x}) = 0$, that is, the two probabilities can not attain the nominal level $1-\alpha$. The numerical simulations illustrate the correctness of Theorems 2.1 and 2.2. In summary, we can not blindly assume that a $1-\alpha$ credible interval will always be a $1-\alpha$ confidence interval, or vice versa. The choice of prior matters.

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Competing Interests

Authors have declared that no competing interests exist.

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