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Aspects of the Fourier-Stieltjes Transform of C^*-algebra Valued Measures

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

This paper deals with the Fourier-Stieltjes transform of C^*-algebra valued measures. We construct an involution on the space of such measures, define their Fourier-Stieltjes transform and derive a convolution theorem.

Keywords: C^*-algebra; vector measure; Fourier-Stieltjes transform; convolution.

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1 Introduction

Banach space valued measures play an important rôle in the geometric theory of Banach spaces. For instance in [1] the author used the theory of vector measures to prove that $L^1[0,1]$ is not isomorphic to a dual of a Banach space. See [2] for interesting historical notes. It is natural to think that C^*-algebra valued measures may be useful in the theory of C^*-algebras. This paper is in some manner a contribution in that direction. Here we are interested in the bounded C^*-algebra valued measures and their Fourier-Stieltjes transform.

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The rest of the paper is structured as follows. In Section 2, we present basic elements of the theory of C^*-algebras with examples. In Section 3, we construct an involution on the space of bounded C^*-algebra valued measures on a locally compact group and finally in Section 4, we defined the Fourier-Stieltjes transform and we prove a convolution theorem.

2 C^*-algebras: Definition and Examples

In this section, we recall what is a C^*-algebra and we give various examples. Interested readers can consult [3, 4]. All the vector spaces considered here are complex vector spaces.

Definition 2.1. A Banach algebra is a Banach space $\mathfrak{A}$ which is also an algebra such that

$$\forall a, b \in \mathfrak{A},\ \|ab\| \leq \|a\|\|b\|. \tag{2.1}$$

Definition 2.2. An involution on an algebra $\mathfrak{A}$ is a map $* : \mathfrak{A} \longrightarrow \mathfrak{A}$ such that

$$\begin{aligned} (a^*)^* &= a, \\ (a+b)^* &= a^* + b^*, \\ (ab)^* &= b^* a^*, \\ (\lambda a)^* &= \bar{\lambda} a^*. \end{aligned}$$

for a, $b \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$. A $*$-Banach algebra is a Banach algebra with an involution.

Definition 2.3. A C^*-algebra is a $*$-Banach algebra $\mathfrak{A}$ such that for all $a \in \mathfrak{A}$,

$$\|a^* a\| = \|a\|^2. \tag{2.2}$$

The following result is well known as the "C^*-condition".

Proposition 2.1. *A $*$-Banach algebra $\mathfrak{A}$ in which $\forall a \in \mathfrak{A}$, $\|a\|^2 \leq \|a^* a\|$ is a C^*-algebra.*

Let us give some examples of C^*-algebras.

Example 2.1.

1. The set of complex numbers $\mathbb{C}$ is the prototype of C^*-algebras. The norm is the modulus $|z|$ and the $*$ operation is the conjugation $\overline{z}$.
2. Let $\mathcal{H}$ be a complex Hilbert space. Denote by $B(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$. Then $B(\mathcal{H})$ is a C^*-algebra under the norm

$$\|T\| = \sup\{\|T\xi\| \,:\, \|\xi\| \leq 1\}$$

and the involution $T \to T^*$ where T^* is the adjoint of T defined by

$$\forall \xi, \eta \in \mathcal{H},\ \langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle.$$

3. Let $M_n(\mathbb{C})$ be the set of square complex matrices of order n. It is a C^*-algebra under the matrix operations, the norm defined by

$$\|A\| = (\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2)^{\frac{1}{2}}$$

where A is the matrix $A = (a_{ij})_{1\leq i\leq n, 1\leq j\leq n}$, and the $*$-operation $A^* = {}^t\overline{A}$.

4. Let X be a compact Hausdorff space. Consider $C(X)$ the set of complex continous functions on X. Then $C(X)$ is a C^*-algebra under the usual pointwise operations on $C(X)$, the norm defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

and the $*$-operation

$$f^*(x) = \overline{f(x)}.$$

Now for a locally compact Hausdorff space X one may consider the set $C_0(X)$ instead of $C(X)$ where $C_0(X)$ is the set of complex continuous functons on X that vanish at infinity. Then $C_0(X)$ is a C^*-algebra under the same operations, the same norm and the same involution as $C(X)$.

3 A $*$-Banach Algebra Structure on $\mathcal{M}^1(G, \mathfrak{A})$

Here we would like to trace how far the C^* algebraic structure can infer the structure of the space of vector measures on a locally compact group G. Let G be a locally compact group and let $\mathfrak{A}$ be a C^*-algebra. We denote by $\mathcal{B}(G)$ the σ-field of Borel subsets of G. Following [2] we call a vector measure any set function $m : \mathcal{B}(G) \to \mathfrak{A}$ such that for any sequence $(A_n)_{n\geq 1}$ of pairwise disjoint elements of $\mathcal{B}(G)$ one has

$$m(\overset{\infty}{\underset{n=1}{}} A_n) = \sum_{n=1}^{\infty} m(A_n). \tag{3.1}$$

A vector measure m is said to be bounded if there exists $M > 0$ such that

$$\forall A \in \mathcal{B}(G),\ \|m(A)\| \leq M.$$

The set of such bounded vector measures is denoted by $\mathcal{M}^1(G, \mathfrak{A})$. The *variation* of a vector measure m is the set function $|m|$ defined by

$$|m|(A) = \sup_\pi \sum_n \|m(A_n)\|,$$

where the supremum is taken over all the partitions π of A into pairewise disjoint measurable subsets of A. If $|m|(G) < \infty$ then m is called a vector measure of bounded variation. To be concrete let us give an example of a vector measure taken from [2] and adapted to the case of a locally compact group.

Example 3.1. We take $G = \mathbb{R}^d$ and we obviously denote by $L^1(\mathbb{R}^d)$ and $\mathcal{C}_0(\mathbb{R}^d)$ the Lebesgue space of complex integrable functions on $\mathbb{R}^d$ and the space of complex continous functions on $\mathbb{R}^d$ which vanish at infinity respectively. The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is

$$\mathcal{F}f(x) := \widehat{f}(x) := \int_{\mathbb{R}^d} f(t)e^{-i\langle x,t\rangle} dt,\ x \in \mathbb{R}^d. \tag{3.2}$$

The function $\widehat{f}$ is a member of $\mathcal{C}_0(\mathbb{R}^d)$ and

$$\|\widehat{f}\|_\infty \leq \|f\|_1. \tag{3.3}$$

Now let $T : L^1(\mathbb{R}^d) \to \mathcal{C}_0(\mathbb{R}^d)$ be a bounded linear operator. A concrete example for T is for instance the Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$. Define

$$m(A) = T(\chi_A) \tag{3.4}$$

where A is a member of the Borel σ-algebra of G. Then $\|m(A)\|_\infty \leq \|T\|\mu(A)$ where μ is the Lebesgue measure of $\mathbb{R}^d$. First notice that m is finitely additive. In fact if A and B are disjoint measurable sets then

$$m(A \cup B) = T(\chi_{A\cup B}) = T(\chi_A + \chi_B) = T(\chi_A) + T(\chi_B) = m(A) + m(B). \tag{3.5}$$

Therefore, for a sequence $(A_n)_{n\geq 1}$ of pairwise disjoint measurable sets we have

$$\begin{aligned} \|m({}_{n=1}^{\infty} A_n) - \sum_{n=1}^{k} m(A_n)\| &= \|m({}_{n=1}^{k} A_n) + m({}_{n=k+1}^{\infty} A_n) - \sum_{n=1}^{k} m(A_n)\| \\ &= \|m({}_{n=k+1}^{\infty} A_n)\| \\ &\leq \|T\|\mu({}_{n=k+1}^{\infty} A_n) \\ &= \|T\| \sum_{n=k+1}^{\infty} \mu(A_n) \to 0 \text{ when } k \to \infty \end{aligned}$$

since the real series $\sum_n \mu(A_n)$ is convergent and therefore the remainder $\sum_{n=k+1}^{\infty} \mu(A_n)$ goes to 0 whenever k tends to ∞. We conclude that m is a vector measure taking values in the C^*-algebra $\mathcal{C}_0(\mathbb{R}^d)$.

To move forward, we present some properties of $\mathcal{M}^1(G, \mathfrak{A})$.

On $\mathcal{M}^1(G, \mathfrak{A})$, one defines the norm:

$$\|m\| = |m|(G) \tag{3.6}$$

and the convolution product

$$m_1 * m_2(f) = \int_G \int_G f(xy)dm_1(x)dm_2(y), \tag{3.7}$$

where $m_1, m_2 \in \mathcal{M}^1(G, \mathfrak{A})$ and $f \in \mathcal{C}_0(G, \mathfrak{A})$. And one has

$$\|m_1 * m_2\| \leq \|m_1\|\|m_2\|.$$

It is well-known that $\big(\mathcal{M}^1(G, \mathfrak{A}), \|\cdot\|, *\big)$ is a Banach algebra.

Proposition 3.1. *If $\mathfrak{A}$ is unital then so is $\mathcal{M}^1(G, \mathfrak{A})$.*

Proof. Let us assume that $\mathfrak{A}$ has a unit $1_{\mathfrak{A}}$. For $A \in \mathcal{B}(G)$, set

$$\Delta(A) = \delta(A)1_{\mathfrak{A}} = \begin{cases} 1_{\mathfrak{A}} & \text{if} \quad e \in A \\ 0 & \text{otherwise} \end{cases}$$

where δ is the Dirac mass at e (the neutral element in the group G). It follows that

$$\Delta * m(f) = \int_G \int_G f(xy)d\Delta(x)dm(y) = \int_G f(y)dm(y) = m(f),$$

that is $\Delta * m = m$. We have also

$$m * \Delta(f) = \int_G \int_G f(xy)dm(x)d\Delta(y) = \int_G f(x)dm(x) = m(f),$$

that is $m * \Delta = m$. Hence Δ is the unit of $\mathcal{M}^1(G, \mathfrak{A})$. □

Proposition 3.2. $\mathcal{M}^1(G, \mathfrak{A})$ *is an involutive Banach algebra.*

Proof. We know already that $\mathcal{M}^1(G,\mathfrak{A})$ is a Banach algebra. On this algebra, let us now define an involution. For $m \in \mathcal{M}^1(G,\mathfrak{A})$, set

$$m^{\blacktriangle}(A) = m(A^{-1})^*, \; \forall A \in \mathcal{B}(G). \tag{3.8}$$

where $A^{-1} = \{x^{-1} : x \in A\}$, or equivalently

$$m^{\blacktriangle}(f) = \int_G f(x^{-1}) dm^*(x) \tag{3.9}$$

where $*$ is the involution of the C^*-algebra $\mathfrak{A}$ and f belongs to $\mathcal{C}_c(G;\mathfrak{A})$, the space of $\mathfrak{A}$-valued functions with compact support. One can easily check that the mapping $m \mapsto m^{\blacktriangle}$ defines an involution on $\mathcal{M}^1(G,\mathfrak{A})$.

□

4 The Fourier-Stieltjes Transform

Research on the Fourier-Stieltjes transform stays flourishing. A recent study concerning this subject can be found in [5]. Our analysis here borrows ideas from [6, 7, 8, 9]. Methods there were applied to the case where G is a compact group or G acts on a finite dimensional Hilbert C^*-module. With a little adaptation we applied it to the case of a general locally compact group. For more informations about representation theory and Fourier analysis on groups, on may consult [10, 11, 12].

There are various formulations of the Fourier-Stieltjes transform depending on the nature of the underlying group and the structure of the codomain of the measures.

In the case G is abelian, the Fourier-Stieltjes transform of the vector measure m is

$$\widehat{m}(\chi) = \int_G \overline{\langle \chi, x\rangle} dm(x), \tag{4.1}$$

where χ designates a character of the group G. If G is compact and $\mathfrak{A} = \mathbb{C}$, then the Fourier-Stieltjes transform of m is a family $(\widehat{m}(\sigma))_{\sigma\in\widehat{G}}$ of endomorphisms $\widehat{m}(\sigma) : \mathcal{H}_\sigma \to \mathcal{H}_\sigma$ given by the relation:

$$\langle \widehat{m}(\sigma)\xi, \eta\rangle = \int_G \langle \sigma(x^{-1})\xi, \eta\rangle dm(x), \; \xi, \eta \in \mathcal{H}_\sigma. \tag{4.2}$$

where σ is a member of a class of unitary irreducible representation of G, $\mathcal{H}_\sigma$ is the representation space of σ and $\widehat{G}$ is the unitary dual of G. When the group G is compact and $\mathfrak{A}$ is a Banach space, the Fourier-Stieltjes transform of a bounded vector measure m on G is defined and studied in [6]. It is interpreted as a family $(\widehat{m}(\sigma))_{\sigma\in\widehat{G}}$ of sesquilinear mappings $\widehat{m}(\sigma) : \mathcal{H}_\sigma \times \mathcal{H}_\sigma \to \mathfrak{A}$ given by:

$$\widehat{m}(\sigma)(\xi, \eta) = \int_G \langle \sigma(x^{-1})\xi, \eta\rangle dm(x). \tag{4.3}$$

We denote the conjugate space of $\mathcal{H}_\sigma$ by $\overline{\mathcal{H}}_\sigma$. We denote by $\mathcal{H}_\sigma \hat{\otimes}_\pi \overline{\mathcal{H}}_\sigma$ the completion of the normed tensor product space $\mathcal{H}_\sigma \otimes \overline{\mathcal{H}}_\sigma$ with respect to the projective tensor norm π. See [13] for more informations on the tensor product of Banach spaces.

Let m be a vector measure on a locally compact group G. From [8] we see that the Fourier-Stieltjes transform of m is the collection $(\widehat{m}(\sigma))_{\sigma\in\widehat{G}}$ of operators $\widehat{m}(\sigma) : \mathcal{H}_\sigma \hat{\otimes}_\pi \overline{\mathcal{H}}_\sigma \to \mathfrak{A}$ where each $\hat{m}(\sigma)$ is defined by the integral

$$\widehat{m}(\sigma)(\xi \otimes \eta) = \int_G \langle \sigma(x^{-1})\xi, \eta\rangle dm(x). \tag{4.4}$$

We denote by $\mathcal{L}(\mathcal{H}_\sigma \hat{\otimes}_\pi \overline{\mathcal{H}}_\sigma, \mathfrak{A})$ the set of bounded operators from $\mathcal{H}_\sigma \hat{\otimes}_\pi \overline{\mathcal{H}}_\sigma$ into $\mathfrak{A}$.

Example 4.1. Consider the matrix group $G = SU(2)$ where

$$SU(2) = \{A \in M_2(\mathbb{C}) \,:\, A^*A = I,\, \det A = 1\}$$
$$= \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \,:\, a, b \in \mathbb{C},\, |a|^2 + |b|^2 = 1 \right\}.$$

Let H_2 be the set of homogeneous polynomials of degree 2 in two variables z_1, z_2. Then

$$H_2 = \mathbb{C}z_1^2 \oplus \mathbb{C}z_1z_2 \oplus \mathbb{C}z_2^2.$$

Now consider the representation $\sigma : SU(2) \to GL(H_2)$ given by

$$[\sigma(A)f](z_1, z_2) = f((z_1, z_2)A),\ A \in SU(2),\ f \in H_2. \tag{4.5}$$

Consider a bounded linear operator $T : L^1(SU(2)) \to \mathcal{C}_0(SU(2))$ and the vector measure m given by $m(E) = T(\chi_E)$, so that $m(f) = Tf$ for f integrable with respect to the Haar measure on $SU(2)$. Then the Fourier-Stieltjes transform of m is given by

$$\widehat{m}(\sigma)(f \otimes g) = m(\phi^{\sigma}_{f,g}) = T(\phi^{\sigma}_{f,g}) \tag{4.6}$$

where $\phi^{\sigma}_{f,g}(A) = \langle \sigma(A^{-1})f, g\rangle$.

Proposition 4.1. *If $m \in \mathcal{M}^1(G, \mathfrak{A})$ and $\sigma \in \widehat{G}$ then $\widehat{m}(\sigma) \in \mathcal{L}(\mathcal{H}_\sigma \hat{\otimes}_\pi \overline{\mathcal{H}}_\sigma, \mathfrak{A})$ and $\|\widehat{m}(\sigma)\|_{\mathcal{H}_\sigma \hat{\otimes}_\pi \overline{\mathcal{H}}_\sigma \to \mathfrak{A}} \leq \|m\|$.*

Proof. Let $m \in \mathcal{M}^1(G, \mathfrak{A})$. For each $\sigma \in \widehat{G}$, we have

$$\begin{aligned}\|\widehat{m}(\sigma)(\xi \otimes \eta)\| &= \|\int_G \langle \sigma(x^{-1})\xi, \eta\rangle dm(x)\| \\ &\leq \int_G \|\langle \sigma(x^{-1})\xi, \eta\rangle\| d|m|(x) \\ &\leq \|\xi\|\|\eta\||m|(G) = \|\xi\|\|\eta\|\|m\|.\end{aligned}$$

Thus $\widehat{m}(\sigma)$ is a bounded operator and $\|\widehat{m}(\sigma)\|_{\mathcal{H}_\sigma \hat{\otimes}_\pi \overline{\mathcal{H}}_\sigma \to \mathfrak{A}} \leq \|m\|$. ☐

Using arguments form [7, Lemma 4.1.5] applied to the underlying Banach space structure of $\mathfrak{A}$, one obtains the injectivity of the Fourier-Stieltjes transform $m \mapsto \widehat{m}$.

Proposition 4.2. *The map $m \mapsto \widehat{m}$ from $\mathcal{M}^1(G, \mathfrak{A})$ into $\prod_{\sigma \in \widehat{G}} \mathcal{L}(\mathcal{H}_\sigma \hat{\otimes}_\pi \overline{\mathcal{H}}_\sigma, \mathfrak{A})$ is injective.*

Proposition 4.3. *If $m \in \mathcal{M}^1(G, \mathfrak{A})$ and $T \in \mathcal{L}(\mathcal{H}_\sigma \hat{\otimes}_\pi \overline{\mathcal{H}}_\sigma, \mathfrak{A})$ then the mapping*

$$x \mapsto T[(\sigma(x^{-1})\xi) \otimes \eta]$$

from G into $\mathfrak{A}$ is integrable with respect to m.

Proof.

$$\begin{aligned}\int_G \|T[(\sigma(x^{-1})\xi) \otimes \eta]\| dm(x) \leq& \|T\|\|\xi\|\|\eta\| \int_G \chi_G d|m| \\ =& \|T\|\|\xi\|\|\eta\|\|m\| < \infty.\end{aligned}$$

Thus the map $x \mapsto T[(\sigma(x^{-1})\xi) \otimes \eta]$ is m-integrable. ☐

For $T \in \mathcal{L}(\mathcal{H}_\sigma \otimes \overline{\mathcal{H}}_\sigma, \mathfrak{A})$ and $m \in \mathcal{M}^1(G, \mathfrak{A})$, one defines the product $\sharp$ by:

$$T\sharp[\widehat{m}(\sigma)](\xi \otimes \eta) = \int_G T[(\sigma(x^{-1})\xi) \otimes \eta]dm(x). \tag{4.7}$$

Then we have the following analog of the well-known convolution theorem.

Proposition 4.4. *If $m, n \in \mathcal{M}^1(G, \mathfrak{A})$ then*

$$(\widehat{n * m})(\sigma) = \widehat{m}(\sigma)\sharp\widehat{n}(\sigma). \tag{4.8}$$

Proof. Let m and n be in $\mathcal{M}^1(G, \mathfrak{A})$ and $\xi \otimes \eta \in \mathcal{H}_\sigma \otimes \mathcal{H}_\sigma$. We have:

$$\begin{aligned}
[\widehat{m}(\sigma)\sharp\widehat{n}(\sigma)](\xi \otimes \eta) &= \int_G \widehat{m}(\sigma)[(\sigma(y^{-1})\xi) \otimes \eta]dn(y) \\
&= \int_G \int_G \langle \sigma(x^{-1})\sigma(y^{-1})\xi, \eta \rangle dm(x)dn(y) \\
&= \int_G \int_G \langle \sigma(x^{-1}y^{-1})\xi, \eta \rangle dm(x)dn(y) \\
&= \int_G \int_G \langle \sigma((yx)^{-1})\xi, \eta \rangle dn(y)dm(x) \text{ (Fubini)} \\
&= \widehat{n * m}(\sigma)(\xi \otimes \eta).
\end{aligned}$$

Hence

$$\widehat{m}(\sigma)\sharp\widehat{n}(\sigma) = (\widehat{n * m})(\sigma).$$

□

Remark 4.1. One knows that the convolution product is commutative if and only if the group G is commutative. Thus if G is commutative we have

$$\widehat{m}(\sigma)\sharp\widehat{n}(\sigma) = (\widehat{n * m})(\sigma) = (\widehat{m * n})(\sigma).$$

5 Conclusion

In this study, we have constructed an involution on the space of bounded measures on a locally compact group taking values in a C^*-algebra. The Fourier-Stieltjes transform of a C^*-algebra valued measure has been defined and finally a convolution theorem has been proved.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Gel'fand IM. Abstrakte funktionen and lineare operatoren. Mat. Sb. (N. S.). 1938;4(46):235-286.

[2] Diestel J, Uhl J. J. JR. . Vector measures. Amer. Math. Soc, Provide. 1977;15.

[3] Averson W. An invitation to C^*-algebras. Springer-Verlag, New York; 1976.

[4] Landsman NP. Lecture notes on c^*-algebras. Hilbert C^*-Modules and Quantum Mechanics; 1998. arXiv: math-ph/9807030v1.

[5] Farashahi AG. Fourier-Stieltjes transforms over homogeneous spaces of compact groups. Groups, Geom. Dyn. 2019;13:511-547.

[6] Assiamoua VSK. Fourier-Stieltjes transforms of vector-valued measures on compact groups. Acta Sci. Math.(Szeged). 1989;53:301-307.

[7] Assiamoua VSK. $L_1(G, A)$-multipliers. Acta Sci. Math.(Szeged). 1989;53:309-3018.

[8] Mensah Y. Facts about the Fourier-Stieltjes transform of vector valued measures on compact groups. Int. J. Anal Appl. 2013;2(1):19-25.

[9] Wodome K, Mensah Y. On a transform of Fourier-Stieltjes type for C^*-algebra valued measures. Pure Math. Sci. 2017;6(1):113-121.

[10] Deitmar A, Echterhoff S. Principles of harmonics analysis. Springer, New York; 2009.

[11] Folland GB. A course in abstract harmonic analysis. CRC Press; 1995.

[12] Gaal SA. Linear analysis and representation theory. Springer, Berlin; 1973.

[13] Ryan RA. Introduction to tensor products of Banach spaces. Monographs in Mathematics, Springer; 2002.

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