



Symmetric Spaces in Riemannian and Semi-Riemannian Geometry

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, we will obtain the necessary and sufficient conditions for the analysis of the position of local symmetry on an arbitrary Riemannian manifold. These conditions are devoid of the aspects of Lie groups, and thus can be used in calculations of procedures, without interfering with the concepts of Lie groups, and improve intuitive attitudes. Also, we will study and create equivalent conditions for a situation where a two-metric homogeneous Riemannian manifold is located symmetrically. In addition, in this paper it is stated that the symmetric space (M, g) can be seen as a homogeneous space G/K . Also, one-to-one correspondence between the symmetric space and the symmetric pair is shown, and curvature is studied on a symmetric space.

Keywords: Symmetric space; Riemannian symmetric space; Semi-Riemannian symmetric space.

1 Introduction

For physical reasons, space-time structures are defined as smooth and connected four-dimensional Lorentzian¹ manifolds. That is, the metric g has a sign $(1, 3)$. The study of this category of metrics is in the

¹ Hendrikus Albertus Lorentz (1853-1928)

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category of semi-Riemannian metrics, that is, unlike the Riemannian metrics, the condition of positivity is not obligatory. In other words, semi-Riemannian metrics are indefinite [1].

The study of semi-Riemannian metrics has many similarities to Riemannian metrics. Therefore, some of the properties can be extended from the Riemannian manifolds to the semi-Riemannian manifolds such as the presence of geodesics and space forms with constant sectional curvatures. However, there are many differences in the study of indefinite metrics. For example, each self-adjoint operator, such as Ricci² operator, is diagonalizable in the Riemannian geometry, while this feature is not necessarily established for semi-Riemannian metrics. Some of the features can be studied for semi-Riemannian metrics without the possibility of examining their matches for Riemannian metrics, such as extending of shear bending study to degenerate plates, the presence of degenerate distributions or the presence of non-diagonalizable operators. In fact, the presence of Jacobi³ degenerate operators results in asymmetric Osserman semi-Riemannian manifolds, or the presence of asymmetric complete complex Einstein hyper surface processes on indefinite flat spaces is also justified in this way [2-5].

An essential feature regarding semi-Riemannian metrics is lack of one-to-one correspondence to the Levi-Civita connection. We know that correspondence to each metric (Riemannian, semi-Riemannian) there is a unique Levi-Civita connection. If the manifold does not have a local de Rham decomposition, the opposite of this is established for the Riemannian metrics to multiplication limit of a constant value. But this correspondence is not established for semi-Riemannian metrics, that is, due to the presence of parallel degenerate vector field there are metrics that define the same Levi-Civita connection, but they are not a constant multiple of the initial metric. It should be noted here that, unlike the Riemannian metrics, due to the indefiniteness of the semi-Riemannian metrics, the presence of parallel degenerate distributions does not result in local reducibility [6].

A semi-Riemannian manifold that has a parallel degenerate distribution is called Walker⁴ manifold. This type of metrics was first studied by Arthur Geoffrey Walker in 1950. Any Walker metric can be written to the standard form using a suitable local coordinate [7].

2 Riemannian Symmetric Space

We study the Riemannian state on symmetric space, which ends up the type as follows [8].

Definition:

A Riemannian symmetric space is a S Riemann manifold with the property that reflectional geodesic at each point is an isometry of S. Otherwise, for every $x \in S$ there is $S_x \in G = I(S)$ (isometric group of S) with these properties:

$$S_x(x) = x, (dS_x)_x = -I$$

Which is called isometry of S_x in symmetric x .

First, the result of this definition: S is generally geodesic. If we define geodesic γ on $[0, S)$, we want to reverse $S_\gamma(t)$ by $t \in (S/2, S)$. So. We want to extend S, in addition S is homogenous.

²Gregorio Ricci-Cubastro (1853-1925)

³Carl Gustav Jacob Jacobi (1804-1851)

⁴Arthur Geoffrey Walker (1909-2001)

For both points of $p, q \in M$ there is an isometry so that the mapping P is on q , in fact the connection of q, P is by the line segment of geodesic γ (because S is complete) and suppose $m \in \gamma$, then $S_m(p) = q$. So G is a transitive action and suppose the point $P \in S$ is a constant basis for it. The subgroup of package $G_P = \{g \in G; g(p) = p\}$ is called isotropic group which is shown with K . The differential at the point P for each $k \in K$ is a step orthogonal conversion of $T_P S$. The determinant of isometry K is by the differential dK_P . Then we want to show K is subgroup of a package of $O(T_P S)$ (the step orthogonal group is on $T_P S$) and is an embedding of $K \mapsto dK_P$ that is called isotropy play, K is compressed in this section. If S is the homogenous space, the isotropy group is an orthogonal action then S is symmetric. If and only of if there is one symmetry S_P for $P \in S$.

Definition:

A Riemannian symmetric space of $M = G / H$ is compressed if *killing* from B to G is indefinite and it is non-compressed, if the infinite negative B is on b and the definite positive is on M .

Theorem:

Suppose $M = G / K$ is a Riemannian symmetric space.

- 1) If M is compressed, then $K \geq 0$ and $Ric \succ 0$ because M is compressed and $\pi_1(M)$ is finite.
- 2) If M is non-compressed, then $K \leq 0$ and $Ric \prec 0$ because M is diffeomorphic of the Euclidean space R^n (which is simple connected). In addition, G is a diffeomorphic of $H \times R^n$.

Proof:

- 1) Myers theorem can be used for a. in fact, M is complete and since Ric is definite positive and maintains $Ric(u, u) \geq a \succ 0$ on a same circle in some of $T_p(M)$.
- 2) For the second part using a proposition (a Riemannian homogenous manifold which has simple connected $Ric \prec 0$ and $K \leq 0$) which shows M is simply connected. Therefore, using Hadamards theorem, M is a diffeomorphic of R^n and also it has pole. Therefore this lemma shows this claim about G . We have to calculate the curvature only in cases where the tensiometer in M is caused by internal multiplication of $cB \downarrow m$. where B is a Killing of G and C is constant so that $C \succ 0$, if M is non-compressed. But $C \succ 0$, if M is non-compressed.

In the general state that is used in linear algebra types in this case:

$$K(X, Y) = CB([X, Y], [X, Y]) / Q(X, Y) \text{ for } X, Y \in M$$

Then $[X, Y] \in \tau$, so considering the definition it results that in the compressed state $K \geq 0$ and in the non-compressed state $K \leq 0$.

Since M is Riemannian and following that $Ric \geq 0$ and $Ric \leq 0$, respectively. Therefore, what remain shows $Ric(X, X) = 0$ and this implies that $X = 0$.

And suppose E_1, \dots, E_n, X is the orthonormal for m that does not change in any state of K because:

$$\begin{aligned} \text{for all } i, \quad Ric(X, X) = 0 &\Rightarrow K(X, E_i) = 0 \\ &\Rightarrow B([X, E_i], [X, E_i]) = 0 \quad \text{for all } i, \\ &\Rightarrow B([X, Y], [X, Y]) = 0 \quad \text{for all } Y \in m, \end{aligned}$$

Because $cB \downarrow_m$ is the internal multiplication and following this $[X, Y] = 0$ and $ad_x(m) = 0$ and since $ad_x ad_x(\tau) = ad_x[X, \tau] \subset ad_x(m) = 0$ and this results us and by definition of *killing* the form $B(X, X) = 0$ then $X = 0$.

If M is a Riemannian symmetric space, the identity component G of the isometric group M is a transitive action of Lie group on M (M is Riemannian homogenous).

Therefore, the point P from M is constant and M is a diffeomorphic from submultiple of G / K where K is the isotopic group the action G on M at point P. An isometric action of K on $T_p(M)$ is obtained by differencing in P. therefore, K is a subgroup of the orthogonal group on $T_p(M)$ so it is compressed.

In addition, we show S_p as $S_p : M \rightarrow M$ which is symmetric to geodesic of M at the point P that the mapping $h \mapsto S_p \circ h \circ S_p$ and $\sigma : G \rightarrow G$ is an automorphism extension of Lie group so that the isotopic group K- included is between the constant point of group σ and this is the identity component (so it is an open subgroup).

In short, M is a symmetric space of G / K with the isotopic group component of K. As a result, the symmetric space to the compressed isotopic group is Riemannian symmetric space.

So the structure of Riemannian symmetric space is an internal multiplication of K-constant on tangential space of G / K in the identity eK . So that there is always the average of an internal multiplication and since K is compressed so a constant Riemannian meter of g on G / K is obtained by the action G.

We show that G / K is Riemannian symmetric. Suppose each $p = hk$ and $h \in G$

And we define:

$$S_p : M \rightarrow M, h'K \mapsto h\sigma(h^{-1}h')K$$

Where extended σ is from G of constant K. Then, we fist examine that S_p is an isometry.

In fact, $S_p(P) = P$ and $dS_p = -I$ on $T_p(M)$. therefore, S_p is a symmetric geodesic and since P is arbitrary so M is a Riemannian symmetric space.

3 Spherical Riemannian Symmetric Space

Suppose M is a Riemannian manifold with Riemannian structure of Q ; M is called an analytic Riemannian manifold, if M and Q are both analytic.

Definition:

Suppose M is an analytic Riemannian manifold; M is called spherical Riemannian symmetric, if each $P \in M$ is a constant point separated from an isometric extension of S_p from M [9].

Lemma:

Suppose M is spherical Riemannian symmetric. For each $P \in M$ there is a vertical N_p neighborhood of $P \in M$ so that S_p is geodesic symmetric on N_p .

Suppose $A = (dS_p)$, then $AM_p \subset M_p$ and $A^2 = I$; we write: and we see that $X = 1/2(X - AX) + 1/2(X + AX)$ (direct summation) where $V^\pm = \{X : AX = \pm X\}$.

Suppose $X \neq 0$ belongs to V^+ , consider a γ geodesic tangential to X , then S_p eliminates the point constant of γ . and this assumption is contradictory that P is a separate constant point. Therefore $A = -1$.

Suppose M is a spherical Riemannian symmetric and γ is any geodesic in M . If $P \in \gamma$ and $S_p \cdot \gamma$ is an extension of γ . also each maximal geodesic in M has as infinite size.

Therefore, M is complete and the both $q \in M$ and P can be a geodesic connection of the size $d(p,q)$. If m is the middle point of the geodesic, then S_m is the convertor of P and Q . in this section of group $I(M)$, the transitive action is on M . where $I(M)$ has a countable basis in the compressed open topology and a transitive and the topologic convertor group is locally compressed M . And suppose K is a subgroup of $I(M)$, then K is compressed as well and $I(M) / K$ is a homomorphism of M under the mapping of $gk \rightarrow g$ and $p_0, g \in I(M)$.

Lemma:

Suppose M is a spherical Riemannian symmetric space, then $I(M)$ is an analytic structure compatible with the compressed open topology of a Lie conversion group from M .

Theorem:

- (i) Suppose M is a spherical Riemannian symmetric space and P_0 is any point in M . if $G = I_0(M)$ and K is a subgroup of G with eliminating the constant P_0 . Then, K is the compressed subgroup of the connected group of G and G / K is analytic diphomorphic to M under the mapping of $gk \rightarrow g$ and $g \in g$ and P_0 .
- (ii) The mapping of $\sigma : g \rightarrow S_{p_0} g S_{p_0}$ is an automorphism extension of G . so that K is also a closed group of (for all constant points of σ and identity components of K_σ). The group $-K$ does not include the natural subgroup of G , otherwise $\{e\}$.
- (iii) Suppose g and ℓ show Lie algebra of G and K , respectively, then $\ell = \{X \in g : (d\sigma)_e X = X\}$ and if $P = \{X \in g : (d\sigma)_e X = -X\}$, we have $g = \ell + P$ (direct summation).

Suppose π shows the natural mapping of $g \rightarrow \mathfrak{g}$; P_0 of G is on M , then $(d\pi)_e$ is the image of ℓ into $\{0\}$ and also P is an isomorphic on M_{P_0} . If $X \in P$, then geodesic caused by P_0 with the tangential vector of $(d\pi)_e X$ is obtained by the following relation.

$$\gamma_{d\pi.X}(t) = \exp tX.P_0 \quad (d\pi = (d\pi)_e)$$

If $Y \in M_{P_0}$, then $(d \exp tX)_{P_0}(Y)$ is the parallel transition of Y along the geodesic.

Propositions:

Suppose (G, K) is pair of Riemannian symmetric space and suppose π shows the natural mapping of G on G/K and $0 = \pi(e)$ and also suppose σ is an analytic extension automorphism of G so that $(K_\sigma)_0 \subset K \subset K_\sigma$. In each Riemannian structure of constant $-G$; Q is on G/K and the manifold of G/K is a spherical Riemannian symmetric space that the symmetric geodesic of S_0 below is true [10]:

$$\tau(\sigma(g)) = S_0 \tau(g) S_0 \quad S_0 \circ \pi = \pi \circ \sigma \quad g \in G$$

Where S_0 is independent of Q selection.

Note:

The we see that the Riemannian connection on G/K is independent of Q selection.

Proof:

Assume σ is an arbitrary analytic extension automorphism of G so that $(K_\sigma)_0 \subset K \subset K_\sigma$.

More simply, we write $d\sigma$ and $d\pi$ instead of $(d\sigma)_e$ and $(d\pi)_e$; suppose g and ℓ show Lie algebra of G and K , as a result $P = \{X \in \mathfrak{g} : d\sigma X = -X\}$. Then $g = \ell + P$ (direction summation)

And if $X \in P$ and $\mathfrak{R} \in K$ then $\sigma(\exp Ad(\mathfrak{R})tX) = \mathfrak{R} \exp(-tX)\mathfrak{R}^{-1}$. Also $d\sigma Ad(\mathfrak{R})X = -Ad(\mathfrak{R})X$. Therefore, P is constant under $Ad_G(K)$. The mapping of $d\pi$ is the image of g on T_0 which is a tangential space to G/K in 0 and the kernel of $d\pi$ is ℓ . As a result, the isomorphism of P on T_0 is changeable with action of K and $\mathfrak{R} \in K, X \in P$; $d\pi.Ad(\mathfrak{R})X = d\tau(\mathfrak{R}).d\pi(X)$.

In fact, this formula is immediately a sequence of the relation below:

$$\pi(\exp Ad(\mathfrak{R})tX) = \pi(\mathfrak{R} \exp tX \mathfrak{R}^{-1}) = \tau(\mathfrak{R})\pi(\exp tX).$$

Because $Ad_G(K)$ is a compressed group in the relevant topology of $GL(\mathfrak{g})$ and there is a degree 2 infinite absolutely positive for B on constant $-P$ under $Ad_G(K)$. Then the form $Q_0 = B \circ (d\pi)^{-1}$ on T_0 is constant under all mappings of $d\tau(\mathfrak{R})$ and $\mathfrak{R} \in k$.

Suppose the symmetric bilinear correspondence is on $T_0 \times T_0$ which is shown by Q_0 . For each $P \in G / K$ the bilinear form of Q_P on $(G / K)_P \times (G / K)_P$ is shown by $X_0, Y_0 \in \tau_0$ and $Q_P(d\tau(g)X_0, d\tau(g)Y_0) = Q_0(X_0, Y_0)$. Where $g \in G$ so that $g.0 = P$. The constancy on B under $Ad_G(K)$ guarantees well-defined Q_P .

Because each $g \in G$ and $\tau(g)$ is an analytic diffeomorphism of G / H which shows $P \rightarrow Q_P$ is a Riemannian analytic structure on G / K under an action of $-G$; and once more each Riemannian analytic structure the constant $-G$ on G / K occurs for a second degree constant on P.

We define the mapping S_0 of G / K on itself by $S_0 \circ \pi = \pi \circ \delta$. Then, S_0 is an automorphism extension of G / K on itself and $(dS_0)_0 = -I$. We see that S_0 is an isometry.

And suppose $g \in G$ and $P = \tau(G).0$ and $X, Y \in (G / K)_P$. Then the vectors $X_0 = d\tau(g^{-1})y$ and $Y_0 = d\tau(g^{-1})y$ belong to T_0 . And the formula $S_0 \circ \pi = \pi \circ \delta$ is true for each $x \in G$ and $S_0 \circ \tau(g)(xk) = \sigma(gx)k = \sigma(g)\sigma(x)k = (\tau(\sigma(g)) \circ S_0)(xk)$ and $S_0 \circ \tau(g) = \tau(\sigma(g)) \circ S_0$ also: therefore:

$$Q(dS_0X, dS_0Y) = Q(dS_0d\tau(g)X_0, dS_0d\tau(g)Y_0)$$

$$= Q(dS_0X_0, dS_0Y_0) = Q(X_0, Y_0) = Q(X, Y)$$

Therefore, S_0 is an isometry and close to 0 it has to be coincident with the symmetric geodesic. For an arbitrary point $P = \tau(g).0$ in G / K ; geodesic is symmetric by $S_P = \tau(g) \circ S_0 \circ \tau(g^{-1})$.

Where this is an isometry and the space G / K is a spherical Riemannian symmetric space and this is the end of proving.

4 Semi-Riemannian symmetric Space

Definition:

A semi-Riemannian symmetric space is a connected semi-Riemannian manifold of M so that for each $P \in M$ there is a unique isometry where $\xi_P : M \rightarrow M$ that the identity differential mapping is on $T_P(M)$. Where the isometry ξ_P is called spherical symmetric of M in P because through P it is an inverse of geodesics and ξ_P is unique for all Ms and is symmetric local geodesic in P so it is isometry.

Theorem:

Suppose M and \overline{M} are complete and connected and local semi-Riemannian symmetric manifold which is simple connected by M; if $L : T_0(M) \rightarrow T_0(\overline{M})$ is a linear isometry which maintain the curvature.

Then there is a unique semi-Riemannian covering mapping where $\phi : M \rightarrow \overline{M}$ so that $d\phi_0 = L$.

Example:

Suppose $S = R^n$ is with Euclidean meter; $x \in R^n$ is symmetric at each point and the reflectional point of $S_x(x + v) = x - v$. The $E(n)$ Euclidean group produced by the transmission is an isometry group where the linear mapping is step orthogonal and the isotopic group from origin O is also the step orthogonal group $O(n)$. Pay attention that the symmetries do not produce the complete isometric group $E(n)$, except for a subgroup that is a sequential extension of the transmission group.

Example:

Suppose $S = S^n$ is a unit sphere in R^{n+1} with standard scalar multiplication. The symmetry in each $x \in S^n$ in the line Rx in R^{n+1} is reflectional and $x_x(y) = -y + 2 \langle y, x \rangle x$; in this mode, the produced symmetry is a complete isometry group with the step orthogonal group $O(n+1)$. where the isotropic group of the last standard step vector $e_{n+1} = (0, \dots, 0, 1)^T$ is equal to $O(n) \subset O(n+1)$

Example:

R^n is symmetric because for each point of P the mapping $P + x \rightarrow P - x$ is an isometry and the sphere S^n is also symmetric.

Example:

$S^n = SO(n+1)/SO(n)$ of unit sphere in R^{n+1} and S^n with symmetry of ξ in $0 = (1, 0, \dots, 0)$ is symmetric by transmission $(t_0, t_1, \dots, t_n) \rightarrow (t_0, -t_1, \dots, -t_n)$.

Lemma:

The semi-Riemannian symmetric space is complete.

Proof:

We show that the geodesic $\gamma : [0, b) \rightarrow M$ is extensible and select b near the range c and suppose ξ is spherical symmetric in $\gamma(c)$ and since ξ is geodesic inverse by $\gamma(c)$ and is again parametrization of $\xi \circ \gamma$ which expresses an extension of γ .

Note:

Local semi-Riemannian symmetric manifold is complete, simple connected and symmetric.

Proof:

At each point $P \in M$ the identity linear isometry on $T_P(M)$ maintains the curvature. Therefore, using the above theorems $M = \overline{M}$ can be obtained. It is a semi-Riemannian covering mapping $\phi : M \rightarrow M$ so that $d\phi_P = -id$ because M is simple connected and ϕ is an isometry.

Lemma:

If ξ is a symmetric sphere of $M = G/H$ in zero the mapping $\sigma : g \rightarrow \xi g \xi$ where $\xi g \xi$ is an automorphism extension of G the set $F = \text{Fix}(\sigma) = \{g \in G : \sigma(g) = g\}$ is constant points of σ that a closed subgroup of G so that $F_0 \subset H \subset F$.

Proof:

Since ξ is extended and $\xi^{-1} = \xi$. Then σ is connection of ξ so σ is an extended automorphism and F is a close subgroup of G . If $h \in H$, then the image of derivative from isometry of $\sigma(h)$ in zero is $d\xi_0 dh_0 d\xi_0$. where it is only dh_0 since $d\xi_0 = -id$ because M is connected and $\sigma(h) = h$ then $H \subset F$.

Now we show that $F_0 \subset H$ because F_0 is connected by attachment B produced by points $\alpha(t)$ from one-parameter subgroup of F_0 . Then it is enough to show $\sigma(t) \in H$ but $\alpha(\alpha(t)) = \alpha(t)$ and because $\alpha(t), \xi$ is inverse ; therefore:

$$\xi(\alpha(t)0) = \alpha(t)\xi(0) = \alpha(t)0 \quad \text{for all } t.$$

Because 0 is the only constant point of symmetry ξ , provided that $\alpha(t)0 = 0$. Therefore, $\alpha(t)$ is a isotopic group H of zero.

Proposition:

Suppose $M = G/H$ is a semi-Riemannian symmetric space, show:

- 1) The geodesic starts from zero with assumptions as follows:
 $\gamma_{d\Pi X}(t) = \alpha(t)0 = \Pi \alpha(t)$ where α is a one-parameter subgroup of $x \in M$.
- 2) The curvature tensor in zero is equal to $R_{XY}Z = d\Pi[[x, y], z]$ where $X, Y, Z \in T_0(M)$ under corresponding with $d\Pi$ to X and $Y, Z \in m$. If x, y their length is non-degenerate, then:

$$K(x, y) = \langle [[X, Y], X], Y \rangle / Q(X, Y)$$

Proof:

Since $M = G/H$ is naturally reduced of Lie sub-space m which results the formula of sectional curvature. Because for $X, Y \in m$ and $[x, y] \in \tau$ now we first study the below multi-linear function:

$$(X, Y, Z, W) \rightarrow \langle [X, Y], Z \rangle, W \rangle$$

Where it is curved on m which is obviously left symmetric in Y, X . where periodic symmetry in Z, Y, X is only with Jacobi identity. As a result, $[X, Y] \in \tau$ because is left symmetry in W, Z so in fact, m with Scalar multiplication is constant on it.

Theorem:

Suppose H is a closed subgroup of a connected Lie group of G and suppose σ is an extension of automorphism G so that $F_0 \subset H \subset F = \text{Fix}(\sigma)$.

Then each tensor meter constant $-G$ on $M = G / H$ results in a semi-Riemannian symmetric space of M so that $\xi \circ \pi = \pi \circ \sigma$ where ξ is local symmetric of M in 0 and the proposition π is of $G \rightarrow M$.

Proof:

- (a) There is a unique function $\xi : M \rightarrow M$ so that $\xi \circ \pi = \pi \circ \sigma$. If $g \in G$, then $\xi(\pi g) = \pi(\sigma g)$ is a fixed definition because $\pi g_1 = \pi g_2$, that is $g_1 H = g_2 H$, and because σ is constant of H ; then $\sigma(g_1)H = \sigma(g_2)H$ where $\pi \sigma g_1 = \pi \sigma g_2$.
- (b) ξ is a diffeomorphism where ξ is a usual smooth derivative for the existence of a local action of submersion π because is extended therefore $\xi^{-1} = \xi$.
- (c) $d\xi_0 = -id$ eventually $\xi(0) = 0$. If $y \in T_0(M)$, $Y \in \mathfrak{g}$ so that $d\sigma(Y) = -Y$ and $d\pi(Y) = y$ then:

$$d\xi(y) = d\xi(d\pi Y) = d\pi(d\sigma Y) = d\pi(-Y) = -y$$

$$(d)\tau_{\sigma g} = \xi \tau_g \xi. \text{ for } g \in G. \text{ inf act, for all } a \in G,$$

$$\xi \tau_g \pi a = \xi \pi(ga) = \pi \sigma(ga) = \pi(\sigma g \cdot \sigma a) = \tau_{\sigma g} \pi(\sigma a) = \tau_{\sigma g} \xi \pi a.$$

- (d) For each tensor meter of constant $-G$ on $\xi; M$ is an isometry, if $v_0 = d\tau_{g^{-1}}(v) \in T_0(M)$ and suppose $v_0 = d\tau_{g^{-1}}(v) \in T_0(M)$. Then using the equation (c), (d) we have:

$$\begin{aligned} < d\xi v, d\xi v > &= < d\xi d\tau_g(v_0), d\xi d\tau_g(v_0) > \\ &= < d\tau_{\sigma g} d\xi(v_0), d\tau_{\sigma g} d\xi(v_0) > = < d\xi(v_0), d\xi(v_0) > \\ &= < -v_0, -v_0 > = < v, v >. \end{aligned}$$

Proof of the theorem is complete observing if a homogenous space has local symmetry ξ at a point of 0 , the other one is at any $p = \tau(0)$, that is, $\tau \xi \tau^{-1}$.

5 Riemannian Symmetric Manifold

Definition:

A Riemannian symmetric manifold of M is called Riemannian symmetric space, if there is a symmetry of S_X for each point of $X \in M$ so that an isometry of M and a neighborhood of N_X from X where X is a unique constant point of S_X in N_X .

Definition:

A connected Riemannian manifold is called symmetric, if there is an isometric dependent of $\sigma_P : M \rightarrow M$ for each $P \in M$ so that i) σ_P^2 is identity extension ii) and P has a separate constant point and there is a neighborhood of U from P so that P is the only constant point of σ_P .

Lemma:

If is a Riemannian manifold and σ_P is an isometric extension so that P is a separate constant point then $\sigma_{P^*}(X_P) = -X_P$ and $\sigma_P(EXP X_P) = EXP(-X_P)$ ($\forall X_P \in T_P(M)$);

Proof:

Because σ_P^2 is identity and true from $(\sigma_{P^*})^2$ on $T_P(M)$ and this means that the value of the characteristic σ_{P^*} on $T_P(M)$ is +1; anyway if the value of the characteristic is +1, then there is vector $X_P \neq 0$ so that $\sigma_{P^*}(X_P) = X_P$.

For each isometry $F : M \rightarrow M$ and $F \circ EXP = EXP \circ F_*$. Because it maintains geodesics, that is, $\sigma_P(EXPtX) = EXPtX$. Therefore, the geodesic around P in the main direction of X_P is a constant point which means that P cannot be a separate point from σ_P . Therefore, +1 is not the value of the characteristic and $\sigma_{P^*} = -I$ where I in fact is identity. Because σ is an isometry and $\sigma_P(EXP X_P) = EXP \sigma_{P^*}(X_P) = EXP(-X_P)$ and this means that we assume σ_P each geodesic around P on itself in its opposite direction.

Note:

Each complete Riemannian manifold of M and the point $P \in M$ can have at most one isometric extension σ_P from P with a separate constant point.

Theorem:

A Riemannian symmetric manifold of M is necessarily complete and if $p, q \in M$, then there is an isometry $-\sigma$ corresponding to $r \in M$ so that $\sigma_r(p) = q$.

Proof:

First we show that M is complete and prove that any geodesic can be an expansion of infinite size. Suppose P(S) is a geodesic radius form the size of S so that we define $0 \leq S < b$. We want to show an extension of the size of $L > b$ and suppose $S_0 = 3/4b$ and if a symmetry of $P(S_0)$.

Supposing that the geodesic P(S) is around another geodesic P(S₀) in the mode of tangential vector of $P(S_0) = (dp/ds)_{S_0}$ that its value is same P(S). Because it has a common tangent from P(S) to P(S₀). Where P(S) is coincident on the range $1/2b < S < b$; then the size of this expansion is $> 3/2b$.

Following this it is simply observed that each $p, q \in M$ is an isometry of M where $M : p \rightarrow q$. In fact, suppose r is a geodesic from p to q, then the isometry σ_r is a geodesic of $p \rightarrow q$ on itself.

Theorem:

Suppose P(t) that ; a geodesic of a symmetric manifold M and τ_c is a dependent isometry for each real number; then $\tau_c(p(t)) = p(t + c)$.

If $X_{P(0)}$ is an element of $T_{P(0)}(M)$, then $X_{P(t)} = \tau_{t*} X_{P(0)}$ is the dependent parallel vector field along P(t) where t is the variable of τ_{t*} where $\tau_{t*} : T_{P(0)}(M) \rightarrow T_{P(t)}(M)$ is parallel transmission along the geodesic.

Note:

Pay attention that if $P_2 = P(C_2), P_1 = P(C_1)$ both points are from one geodesic P(t) where $-\infty < t < \infty$, then the argument $\sigma_{p_2} \circ \sigma_{p_1}(p(t)) = p(t + 2(c_2 - c_1))$ and $(\sigma_{p_2} \circ \sigma_{p_1})_*$ are mappings of each parallel vector field along P(t) to a parallel vector field.

Theorem:

Suppose $M = G$ and the connected Lie group is complete with constant meter and suppose $X_e \in T_e(G)$, then the unique geodesic of P(t) with $P(0) = e$ and $P(0) = X_e$ is exactly the determinant of the one-parameter group by X_e and other geodesics are left (right) coset of one-parameter groups.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Berger M. A panoramic view of Riemannian geometry. Springer Science & Business Media; 2012.

- [2] Bacák M, Bergmann R, Steidl G, Weinmann A. A second order nonsmooth variational model for restoring manifold-valued images, *SIAM J. Scientific Comput.* 2016;38:A567-A597.
- [3] Caulton A. The role of symmetry in the interpretation of physical theories, *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics.* 2015;52:153-162.
- [4] Helgason S. *Differential geometry and symmetric spaces.* Academic Press; 1962.
- [5] Petersen P. *Riemannian geometry.* Springer, New York; 2006.
- [6] Velimirović L, Majhi P, De UC. Almost pseudo-Q-symmetric semi-Riemannian manifolds. *International Journal of Geometric Methods in Modern Physics.* 2018;15(07):1850117.
- [7] Araújo JG, Lima HFD, Santos FRD, Velásquez MAL. Characterizations of complete linear Weingarten spacelike submanifolds in a locally symmetric semi-Riemannian manifold; 2017.
- [8] Mantica CA, Suh YJ. On weakly conformally symmetric pseudo-Riemannian manifolds. *Reviews in Mathematical Physics.* 2017;29(03):1750007.
- [9] Chaubey SK, Yadav SK. Riemannian manifolds admitting a projective semi-symmetric connection. arXiv preprint arXiv:1710.00622; 2017.
- [10] Zengin FO, Tascı AY. Some properties of pseudo conharmonically symmetric Riemannian manifolds. *Differential Geometry-Dynamical Systems.* 2018;20:126-137.

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