



Some Properties and Estimation of Parameters for the Five Parameter Type I Generalized Half Logistic Distribution under Complete Observation

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Authors' contributions

This work was carried out in collaboration among all authors. Author OAB designed the study. Author POA performed the statistical analysis, managed the analysis and wrote the first draft of the manuscript. Authors IS and HOL managed the literature searches and reviewed the manuscript. All authors read and approved the final manuscript.

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Abstract

This paper is a further study of the five parameter type I generalized half logistic distribution. We derived some properties of the distribution. Estimation of the parameters of the distribution under complete observation was studied using the maximum likelihood method. To assess the flexibility of the distribution, it was applied to a real lifetime data and the results when compared to the sub-models showed that the five parameter type I generalized half logistic distribution performed best.

Keywords: Properties; maximum likelihood; five parameter type I generalized half logistic; sub-models.

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1 Introduction

In solving real life situations, the role of probability distributions cannot be over emphasized. Over the years, different distributions have been derived and extensively studied. These distributions have been applied to real life situations in engineering, health, sciences, finances e.t.c. However, in various cases, classical distributions are not suitable for explaining and solving real world phenomena. So, deriving new distributions have been differently studied by the introduction of additional shape parameter(s) to baseline model.

The type I generalized half logistic distribution derived by Olapade [1], a generalization of a form of half logistic distribution obtained by Balakrishna [2] is gradually gaining attention in distribution theory and application to real life situations. Awodutire et al. [3] studied the survival properties of the distribution and Awodutire et al. [4] applied the derived survival model of Awodutire et al. [3] to breast cancer data. A unique property of the type I generalized distribution, as shared by other distributions from the half logistic distribution is the non-decreasing nature of the hazard function by Bello et al. [5].

The type I generalized half logistic distribution has been further generalized. We have the four parameter half logistic distribution of Olapade [6] and the five parameter type I generalized half logistic distribution as derived by Bello et al. [5]. Bello et al. [5] studied the cumulative density function (c.d.f), moments, median, mode and the 100k-percentage point of the five parameter type I generalized half logistic distribution.

This paper tends to further study some properties of the distribution and examine the estimates of the parameters under complete and censored observations. The distribution will be applied to a real lifetime data.

2 Materials and Methods

2.1 A five-parameter type I generalized half-logistic distribution

The probability density function(pdf) of the five parameter type I generalized half logistic distribution is given as

$$f(x; \mu, \sigma, \lambda, \beta, q) = \frac{\beta q (\lambda + \beta)^q e^{-\frac{x-\mu}{\sigma}}}{\sigma (\lambda + \beta e^{-\frac{x-\mu}{\sigma}})^{q+1}}, \quad 0 < x < \infty, \mu > 0, \sigma > 0, \lambda > 0, \beta > 0, q > 0. \quad (1)$$

where μ is the location parameter
 σ is the scale parameter
 λ, β, q are shape parameters.

The distribution has the cumulative distribution function as

$$F(x; \mu, \sigma, \lambda, \beta, q) = 1 - \left[\frac{\lambda + \beta e^{-\frac{x-\mu}{\sigma}}}{\lambda + \beta} \right]^q, \quad 0 < x < \infty, \mu > 0, \sigma > 0, \lambda > 0, \beta > 0, q > 0 \quad (2)$$

If $\beta = 1$, then the five parameter generalized half logistic distribution reduces to the four parameter generalized distribution as obtained by Olapade [6]. Furthermore, if $\lambda = 1$ and $\beta = 1$, then the five parameter type I half logistic distribution reduces to the type I generalized half logistic distribution proposed by Olapade [1]. With $\lambda = 1, q = 1$ and $\beta = 1$, we have the half logistic distribution as derived by Balakrishnan [2].

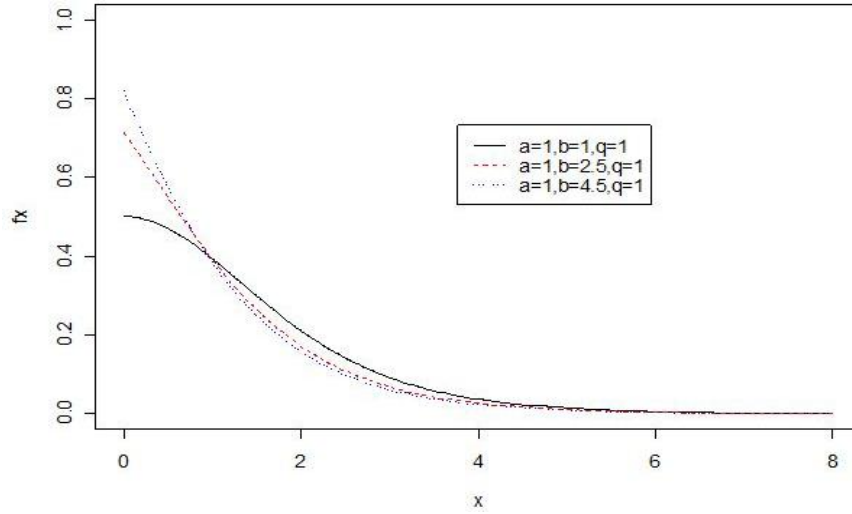


Fig. 1. Graph of the five parameter type I generalized half logistic distribution

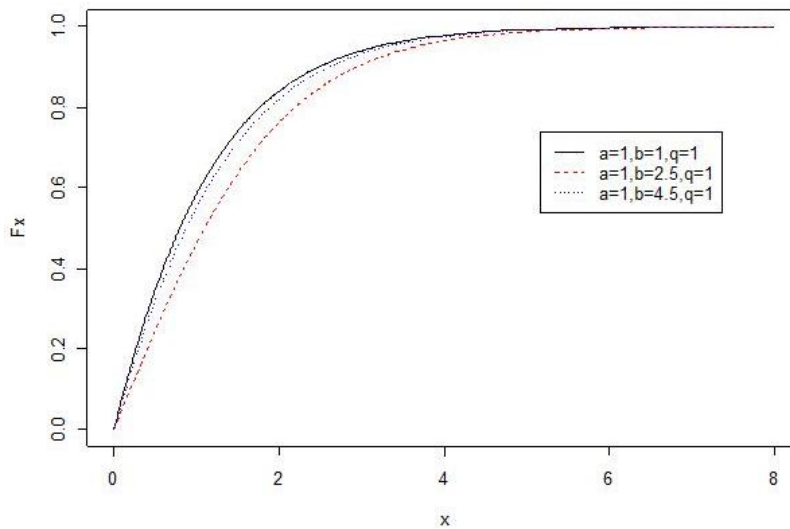


Fig. 2. Graph of the cumulative density function of the five parameter type I generalized half logistic distribution

3 Results and Discussion

3.1 Order statistics of the five-parameter type I generalized half logistic distribution

Let X_1, X_2, \dots, X_n be n independently continuous random variable from the generalized log-logistic distribution and let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ be the corresponding order statistics.

Let $F_{X_{r:n}}(x)$ and $f_{X_{r:n}}(x), r=1,2,3,\dots,n$ denote the cumulative distribution function and the probability density function of the r^{th} order statistics $X_{r:n}$ respectively. The probability density function of $X_{r:n}$ as

$$f_{X_{r:n}} = \frac{1}{B(r, n-r+1)} P^{r-1}(x) [1 - P(x)]^{n-r} p(x). \tag{3}$$

For the five-parameter type I generalized half-logistic distribution with probability density function and cumulative distribution function are given as in equations , the Order Statistics is obtained as

$$f_{X_{r:n}}(x) = \frac{1}{B(r, n-r+1)} \left[1 - \frac{(\lambda+\beta)^q}{(\lambda+\beta e^x)^q} \right]^{r-1} \left[\frac{(\lambda+\beta)^q}{(\lambda+\beta e^x)^q} \right]^{n-r} \frac{\beta q (\lambda+\beta)^q e^x}{(\lambda+\beta e^x)^{q+1}} \tag{4}$$

$$= \frac{1}{B(r, n-r+1)} \frac{\beta q e^x (\lambda+\beta)^{q(n-r+1)}}{(\lambda+\beta e^x)^{qr-q+qn-qr+q+1}} [(\lambda+\beta e^x)^q - (\lambda+\beta)^q]^{r-1}$$

$$= \frac{1}{B(r, n-r+1)} \frac{\beta q e^x (\lambda+\beta)^{q(n-r+1)}}{(\lambda+\beta e^x)^{qn+1}} [(\lambda+\beta e^x)^q - (\lambda+\beta)^q]^{r-1}. \tag{5}$$

3.1.1 The probability density function of the minimum and maximum observations from the five-parameter type I generalized half-logistic distribution

The knowledge of the minimum and maximum observations are of great importance in statistical studies. They are useful in the study of extreme values and range.

3.1.1.1 The Minimum order statistics

Consider the probability density of the r^{th} order statistics from the five-parameter type I generalized half-logistic distribution in equation 5. Let $r = 1$, then the probability density function of the minimum order statistic is

$$f_{X_{1:n}}(x) = \frac{\beta n q (\lambda+\beta)^{qn} e^x}{(\lambda+\beta e^x)^{qn+1}}. \tag{6}$$

The p^{th} moment of the minimum observation from the five-parameter type I generalized half-logistic distribution is

$$E[X_{1:n}^p] = \int x^p f_{X_{1:n}}(x) dx = \beta n q (\lambda+\beta)^{qn} \int_0^\infty \frac{x^p e^x}{(\lambda+\beta e^x)^{qn+1}} dx. \tag{7}$$

The above integral is not close but can be evaluated numerically as we have done for moment of the five-parameter type I generalized half-logistic distribution. So with n known, we can compute the mean, variance, kurtosis and skewness for a set of minimum observation from the five-parameter type I generalized half-logistic distribution.

3.1.1.2 The maximum order statistics

Consider the probability density of the r^{th} order statistics from the five-parameter type I generalized half-logistic distribution in equation (1). Let $r = n$, then the probability density function of the minimum order statistic is

$$f_{X_{n:n}}(x) = \frac{\beta n q e^x (\lambda+\beta)^q}{(\lambda+\beta e^x)^{qn+1}} [(\lambda+\beta e^x)^q - (\lambda+\beta)^q]^{n-1}. \tag{8}$$

Using binomial expansion for positive integral exponents,

$$[(\lambda+\beta e^x)^q - (\lambda+\beta)^q]^{n-1} = \sum_{j=0}^{n-1} (-1)^j (n-1)$$

Therefore

$$\begin{aligned}
 f_{X_{n:m}}(x) &= \frac{\beta n q e^x (\lambda + \beta)^q}{(\lambda + \beta e^x)^{qn+1}} \sum_{j=0}^{n-1} (-1)^j (n-1) \\
 &= \beta n q e^x \sum_{j=0}^{n-1} (-1)^j (n-1) \tag{9}
 \end{aligned}$$

$$\begin{aligned}
 &= \beta n q e^x \sum_{j=0}^{n-1} (-1)^j (n-1) \\
 &= n q \sum_{j=0}^{n-1} (-1)^j (n-1) \\
 &= n q \sum_{j=0}^{n-1} \sum_{k=0}^{qj-qn-1} (-1)^j \lambda^k \beta^{qj-qn-k} (n-1) \tag{10}
 \end{aligned}$$

The p^{th} moment of $X_{n:m}$ is

$$E[X_{n:m}^p] = \theta \int_0^\infty x^p e^{x(qj-qn-k)} dx$$

where $\theta = nq \sum_{j=0}^{n-1} \sum_{k=0}^{qj-qn-1} (-1)^j \lambda^k \beta^{qj-qn-k} (n-1)$

$$E[X_{n:m}^p] = \theta \int_0^\infty x^p e^{-x(qn+k-qj)} dx. \tag{11}$$

Let $x(qn + k - qj) = t$, $x = \frac{t}{(qn+k-qj)}$ and $dx = \frac{dt}{(qn+k-qj)}$

$$\begin{aligned}
 E[X_{n:m}^p] &= \theta \int_0^\infty \left(\frac{t}{qn+k-qj} \right)^p e^{-t} \frac{dt}{qn+k-qj} \\
 &= \frac{\theta}{(qn+k-qj)^{p+1}} \int_0^\infty t^p e^{-t} dt = \frac{\theta \Gamma(p+1)}{(qn+k-qj)^{p+1}} = \frac{p! \theta}{(qn+k-qj)^{p+1}}. \tag{12}
 \end{aligned}$$

Hence, to obtain the first, second, third and fourth moments of the maximum order statistics from the five-parameter type I generalized half-logistic distribution, we use the equation (12) for $p = 1, 2, 3$ and 4 .

When $p = 1$,

$$E[X_{n:m}^1] = \frac{1! \theta}{(qn+k-qj)^{1+1}} = \frac{\theta}{(qn+k-qj)^2} \tag{13}$$

When $p = 2$,

$$E[X_{n:m}^2] = \frac{2! \theta}{(qn+k-qj)^{2+1}} = \frac{2\theta}{(qn+k-qj)^3} \tag{14}$$

When $p = 3$,

$$E[X_{n:m}^3] = \frac{3! \theta}{(qn+k-qj)^{3+1}} = \frac{6\theta}{(qn+k-qj)^4} \tag{15}$$

When $p = 4$,

$$E[X_{n:m}^4] = \frac{4! \theta}{(qn+k-qj)^{4+1}} = \frac{24\theta}{(qn+k-qj)^5} \tag{16}$$

The values of these moments can be used to calculate the mean, variance, skewness and kurtosis of the minimum observations from the five-parameter type I generalized half-logistic distribution.

3.2 Estimation of parameters of the five-parameter type I generalized half-logistic model under complete observation

Given a sample X_1, X_2, \dots, X_n of size n from the five-parameter type I generalized half-logistic distribution with probability density function

$$f(x; \mu, \sigma, \lambda, \beta, q) = \frac{\beta q (\lambda + \beta)^q e^x}{(\lambda + \beta e^x)^{q+1}}, \quad 0 < x < \infty,$$

where μ is the location parameter, σ is the scale parameter, q is the shape parameter, λ and β are the shift parameters. The likelihood function of the generalized log-logistic distribution is obtained as

$$L(x; \mu, \sigma, \lambda, \beta, q) = \frac{\beta^n q^n (\lambda + \beta)^{nq} \exp\left(\sum_{i=1}^n \frac{x_i - \mu}{\sigma}\right)}{\sigma^n \prod_{i=1}^n (\lambda + \beta \exp\left(\frac{x_i - \mu}{\sigma}\right))^{q+1}}. \tag{17}$$

Taking the natural logarithm of both sides in (17), we have

$$\ln L(x; \mu, \sigma, \lambda, \beta, q) = n \ln \beta + n \ln q + n q \ln(\lambda + \beta) + \sum_{i=1}^n \left(\frac{x_i - \mu}{\sigma}\right) - n \ln \sigma - (q + 1) \sum_{i=1}^n \ln[\lambda + \beta \exp\left(\frac{x_i - \mu}{\sigma}\right)] \tag{18}$$

To obtain the estimate of each parameter that maximizes the likelihood function, we differentiate (18) with respect to each of the parameters and equate the derivatives to zero and solve for the parameters.

Hence, by differentiation we have

$$\frac{\partial \ln L(x; \mu, \sigma, \lambda, \beta, q)}{\partial \mu} = -\frac{n}{\sigma} + \frac{\beta(q+1)}{\sigma} \sum_{i=1}^n \left[\frac{e^{\left(\frac{x_i - \mu}{\sigma}\right)}}{\lambda + \beta e^{\left(\frac{x_i - \mu}{\sigma}\right)}} \right]. \tag{19}$$

$$\frac{\partial \ln L(x; \mu, \sigma, \lambda, \beta, q)}{\partial \sigma} = -\frac{n}{\sigma} - \sum_{i=1}^n \frac{(x_i - \mu)}{\sigma^2} + \frac{\beta(q+1)}{\sigma} \sum_{i=1}^n \left[(x_i - \mu) \frac{e^{\left(\frac{x_i - \mu}{\sigma}\right)}}{(\lambda + \beta e^{\left(\frac{x_i - \mu}{\sigma}\right)})} \right]. \tag{20}$$

$$\frac{\partial \ln L(x; \mu, \sigma, \lambda, \beta, q)}{\partial \lambda} = \frac{nq}{\lambda} - (q + 1) \sum_{i=1}^n \frac{1}{(\lambda + \beta e^{\left(\frac{x_i - \mu}{\sigma}\right)})}. \tag{21}$$

$$\frac{\partial \ln L(x; \mu, \sigma, \lambda, \beta, q)}{\partial \beta} = \frac{n}{\beta} + \frac{nq}{\beta} - (q + 1) \sum_{i=1}^n \frac{e^{\left(\frac{x_i - \mu}{\sigma}\right)}}{(\lambda + \beta e^{\left(\frac{x_i - \mu}{\sigma}\right)})}. \tag{22}$$

$$\frac{\partial \ln L(x; \mu, \sigma, \lambda, \beta, q)}{\partial q} = \frac{n}{q} + n \ln(\lambda + \beta) - \sum_{i=1}^n \ln(\lambda + \beta e^{\left(\frac{x_i - \mu}{\sigma}\right)}). \tag{23}$$

Since the equations (19,20,21,22 and 23) above are nonlinear in the parameters and not in closed form, we used numerical iterative method with the aid of R-computer programme to estimate the parameters from a given sample.

3.3 Application of the five parameter type I generalized half logistic distribution to real life data

In this sub-section, we applied the five parameter type I generalized half logistic distribution to a real data set. Data set is given by Murthy et al. [7] on the time between failures for a repairable item. The data set contains 30 observations. The data set are as follows 0.77, 2.63, 3.46, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45,

0.70, 1.49, 1.43, 0.11, 0.71, 1.06, 1.46, 0.30, 1.82, 2.37, 2.46, 0.59, 0.74, 1.23, 0.94, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.

We will fit the five parameter type I generalized half logistic distribution to the data and show flexibility of the distribution in analyzing the data than the other sub-models, that is Olapade [1] and Olapade [6]. For comparison of the models, we employ the Akaike Information Criterion.

Table 1. Table displaying descriptive statistics of failures for repairable items

Minimum	1st Quartile	Median	Mean	3rd Quartile	Maximum	Skewness
0.1100	0.7175	1.2350	1.5427	1.9425	4.7300	1.295462

Table 1 describes the data. Positive value of the skewness showed that the data is rightly skewed, a property that the type I generalized half logistic distribution possesses. This implies that the distribution will be able to handle this data. This is further revealed in Fig. 3.

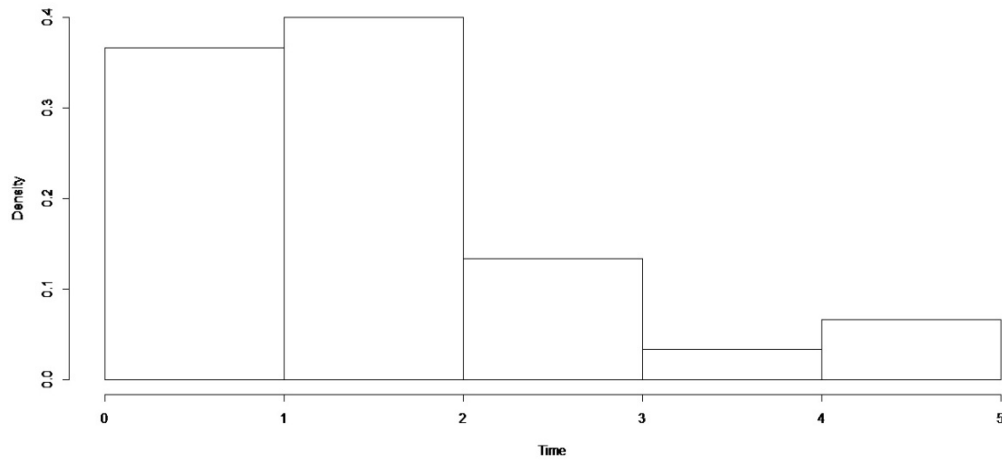


Fig. 3. Histogram displaying the failures for repairable items

Table 2. Results of the application of the models to failure time data

Model	μ	σ	β	λ	q	AIC
Bello et al. (2018)	0.027	0.16	40108	3237	0.1414	38.67
Olapade (2011)	0.02	0.16	10.166	0.1375	-	40.61
Olapade (2014)	0.02	0.16	-	-	0.1130	41.31

From Table 2, it is revealed that the Five Parameter Type I Generalized Half Logistic Distribution performs best due to the lowest value of the AIC.

4 Conclusion

In this paper, we had a further study on the five parameter type I generalized half logistic distribution and obtained its maximum likelihood equations for all the parameters. We applied the distribution to a life time data in which it shows superiority in fitting the data when compare to its sub-models.

Competing Interests

Authors have declared that no competing interests exist.

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