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On the Hyper-Zagreb Index of Some Graph Binary Operations

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Authors' contributions

This work was carried out in collaboration between Both authors. Both authors read and approved the final manuscript.

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Abstract

This study looked at Graph theory as it is an important part of mathematics. Topological indices are numerical parameters of a graph which describe its structure, they have many applications as tools for modeling chemical and other properties of molecules. In this paper, we presented some exact formulas of the Hyper-Zagreb index for some special graphs and some graph binary operations such disjunction $G \vee H$, symmetric difference $G \oplus H$, and tensor product $G \otimes H$ of graphs.

Keywords: Zagreb index; Hyper-Zagreb index; Forgotten index; graph operation.

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1 Introduction

Throughout this paper, we consider a finite connected graph G that has no loops or multiple edges with vertex and edge sets $V(G)$ and $E(G)$, respectively. For a graph G , the degree of a vertex u is the number of edges incident to u, denoted by $\delta(u)$ [1]. Usage of topological indices in chemistry began in 1947 [2] when chemist Harold Wiener developed the most widely known topological descriptor, the Wiener index, and used it to determine physical properties of types of alkanes known as paraffin [3, 4]. In a graph theoretical language the Wiener index is equal to the sum of distances between all pairs of vertices of the respective graph. The first and second Zagreb indices have been introduced by Gutman and Trinajestic in 1972 [5]. They are respectively defined as:

$$\begin{aligned} M_1(G) &= \sum_{v\in V(G)} {\delta_G}^2(v) \\ &= \sum_{uv\in E(G)} \delta_G(u) + \delta_G(v) \end{aligned}$$

$$M_2(G) = \sum_{uv\in E(G)} \delta_G(u)\,\delta_G(v)$$

In 2013, G.H. Shirdel, H. Rezapour and A.M. Sayadi [6] iintroduced distance-based of Zagreb indices named Hyper-Zagreb index which is defined as:

$$HM(G) = \sum_{uv\in E(G)} (\delta_G(u) + \delta_G(v))^2$$

Furtula and Gutman in 2015 introduced forgotten index (F-index) [7, 8, 9] which defined as:

$$F(G) = \sum_{uv\in E(G)} \left({\delta_G}^2(u) + {\delta_G}^2(v)\right)$$

2 Preliminaries

In this section we give some basic and preliminary concepts which we shall use later.

Definition 2.1 :[5]

The disjunction $G \vee H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $u_1\, v_1$ is adjacent with $u_2\, v_2$ whenever $u_1\ u_2 \in E(G)$ or $v_1\ v_2 \in E(H)$.

The symmetric difference $G \oplus H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $E(G \oplus H) = (u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G) or u_2v_2 \in E(H)$ but not both.

The tensor product $G \otimes H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $G \otimes H = (u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G) and u_2v_2 \in E(H)$.

Lemma 2.2:[6, 10] Let G and H be two connected graphs, Then

1. $|V(G \times H)| = |V(G \vee H)| = |V(G \circ H)| = |V(G \oplus H)| = |V(G)||(H)|$,

2. $|E(G \times H)| = |E(G)||V(H)| + |V(G)||E(H)|$,

3. $|E(G+H)| = |E(G)| + |E(H)| + |V(G)||V(H)|$,

4. $|E(G \circ H)| = |E(G)||V(H)|^2 + |E(H)||V(G)|$,

5. $|E(G \vee H)| = |E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 2|E(G)||E(H)|$,

6. $|E(G \oplus H)| = |E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 4|E(G)||E(H)|$,

7. $|E(G \otimes H)| = 2|E(G)||E(H)|$,

8. $\delta_{G \times H}(u,v) = \delta_G(u) + \delta_H(v)$,

9. $\delta_{G \circ H}(u,v) = |V(H)|\delta_G(u) + \delta_H(v)$,

10. $\delta_{G \vee H}(u,v) = |V(H)|\delta_G(u) + |V(G)|\delta_H(v) - \delta_G(u)\delta_H(v)$,

11. $\delta_{G \oplus H}(u,v) = |V(H)|\delta_G(u) + |V(G)|\delta_H(v) - 2\delta_G(u)\delta_H(v)$,

12. $\delta_{G+H}(u,v) = (\delta_G(u) + |V(H)|) if u \in V(G) or (\delta_H(u) + |V(G)|) if u \in V(H)$,

13. $\delta_{G \otimes H}(u,v) = \delta_G(u)\delta_H(v)$.

Example 2.3: Let G_1, G_2 be the atoms of carbon from Chemical compounds: Cyclopropane, Cyclobutane, Cyclopentane and Cyclohexane respectively depicted in Figure 1 Thus,

$$HM(G_1) = \sum_{uv \in E(G_1)} [\delta_{G_1}(u) + \delta_{G_1}(v)]^2 = 48,$$

$$HM(G_2) = 64; HM(G_3) = 80; HM(G_4) = 96;$$

Corollary 2.4:[5] The first and second Zagreb index of some well-known graphs:

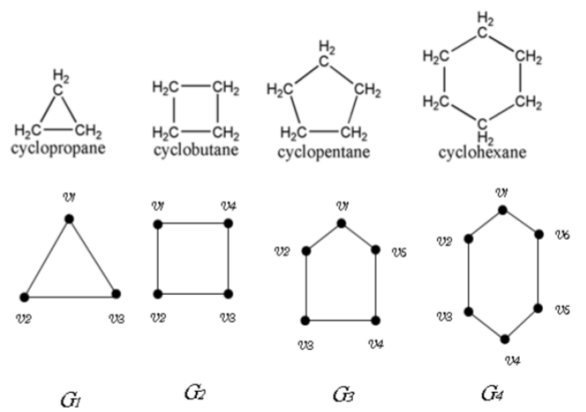

Fig. 1. The atoms of carbon from Chemical compounds

1. For complete graph K_n, with n vertices:

$$M_1(K_n) = n(n-1)^2, M_2(K_n) = (1/2)n(n-1)^3.$$

2. For a cycle graph C_n, with n vertices :

$$M_1(C_n) = 4n, M_2(C_n) = 4n.$$

3. For a path graph P_n, with n vertices :

$$M_1(P_n) = 4n - 6 : n \geq 2, M_2(P_n) = 4(n-2) : n > 2.$$

4. For a path graph and a cycle graph with $m, n, k \geq 3$, vertices :
   1. $M_1(P_n \times C_m) = 16mn - 14m$,
   2. $M_2(P_n \times C_m) = 32mn - 38m$,
   3. $M_1(C_K \times C_m) = 16km$,
   4. $M_2(C_K \times C_m) = 32mk$,

Corollary 2.5:[11, 12] The Hyper-Zagreb index of some well-known graphs:

1. For complete graph K_n, with n vertices:

$$HM(K_n) = 2n(n-1)^3.$$

2. For a cycle graph C_n, with n vertices :

$$HM(C_n) = 16n.$$

3. For a path graph P_n, with n vertices :

$$HM(P_n) = 16n - 30.$$

4. For a path graph and a cycle graph with $m, n, k \geq 3$, vertices :
   1. $HM(P_n \times C_m) = 128nm - 150m$,
   2. $HM(C_K \times C_m) = 128km$,

Theorem 2.6: Let G and H be graphs. Then:

1. $HM(G + H) = HM(G) + HM(H) + 5(|V(G)|M_1(H) + |V(H)|M_1(G)) + 8[|V(G)|^2|E(H)| + |V(H)|^2|E(G)| + |E(G)||E(H)|] + |V(G)||V(H)|[(|V(H)| + |V(G)|)^2 + 4(|E(G)| + |E(H)|)]$.

2. $HM(G \times H) = |V(H)|HM(G) + |V(G)|HM(H) + 12|E(G)|M_1(H) + 12|E(H)|M_1(G)$.

3. $HM(G \circ H) = |V(H)|^4 HM(G) + |V(G)|HM(H) + 12|V(H)|^2|E(H)|M_1(G) + 10|V(H)||E(G)|M_1(H) + 8|E(H)||E(G)|$.

4. $HM(G * H) = HM(G) + |V(G)|HM(H) + 5|V(H)|M_1(G) + 5|V(G)|M_1(H) + 4|V(H)||E(G)|[2|V(H)|+1]+8|E(H)|[|V(G)|+|E(G)|]+|V(G)||V(H)|[|V(H)|^3+2|V(H)|+4|E(H)|]$.

Proof. For the proof (Theorem 2.6) we refer to [6, 11].

3 Main Results

In the following section, we study the Hyper-Zagreb index of some graph operations.

Theorem 3.1: Let G and H be graphs. Then:

$HM(G \vee H) =$
$[|V(G)|^4 - 2|V(H)|^2|E(H)|]HM(H) + [|V(H)|^4 - 2|V(H)|^2|E(H)|]HM(G) + 5|V(G)|M_1(G)F(H) +$
$5|V(H)|M_1(H)F(G) + 10|V(H)|^2|E(H)||V(G)|M_1(G) + 10|V(H)||V(G)|^2|E(G)|M_1(H) +$
$8|V(H)|^2|E(H)||E(G)| + 8|V(G)|^2|E(G)||E(H)| - 8|V(H)||E(G)|^2 M_1(H) -$
$8|V(G)||E(H)|^2 M_1(G) - 4|V(G)|^2|E(G)|F(H) - 4|V(H)|^2|E(H)|F(G) - 8|V(G)|^2|E(G)|M_2(H) -$
$8|V(H)|^2|E(H)|M_2(G) + 8|E(G)|M_2(H) + 8|E(H)|M_2(G) - 8|V(H)||V(G)|M_1(G)M_1(H) +$
$4|V(H)|M_2(G)M_1(H) + 4|V(G)|M_2(H)M_1(G) - 2F(G)F(H) - 4M_2(G)M_2(H)$.

Proof. By Definition 2.1 and Lemma 2.2, we have

$$
\begin{aligned}
HM(G \vee H) = & \\
= & \sum_{(a,c)(b,d)\in E(G\vee H)} [\delta_{(G\vee H)}(a,c) + \delta_{(G\vee H)}(b,d)]^2 \\
= & \sum_{(a,c)(b,d)\in E(G\vee H)} [|V(H)|\delta_G(a) + |V(G)|\delta_H(c) - \delta_G(a)\delta_H(c) + |V(H)|\delta_G(b) \\
+ & |V(G)|\delta_H(d) - \delta_G(b)\delta_H(d)]^2 \\
= & \sum_{ab\in E(G)} \sum_{c\in V(H)} \sum_{d\in V(H)} [|V(H)|\delta_G(a) + |V(G)|\delta_H(c) - \delta_G(a)\delta_H(c) + |V(H)|\delta_G(b) \\
+ & |V(G)|\delta_H(d) - \delta_G(b)\delta_H(d)]^2 \\
+ & \sum_{cd\in E(H)} \sum_{a\in V(G)} \sum_{b\in V(G)} [|V(H)|\delta_G(a) + |V(G)|\delta_H(c) - \delta_G(a)\delta_H(c) + |V(H)|\delta_G(b) \\
+ & |V(G)|\delta_H(d) - \delta_G(b)\delta_H(d)]^2 \\
- & 2\sum_{ab\in E(G)} \sum_{cd\in E(H)} [|V(H)|\delta_G(a) + |V(G)|\delta_H(c) - \delta_G(a)\delta_H(c) + |V(H)|\delta_G(b) \\
+ & |V(G)|\delta_H(d) - \delta_G(b)\delta_H(d)]^2,
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{ab\in E(G)} \sum_{c\in V(H)} \sum_{d\in V(H)} [[|V(H)|(\delta_G(a) + \delta_G(b)) + |V(G)|(\delta_H(c) + \delta_H(d))]^2 \\
- & 2[|V(H)|(\delta_G(a) + \delta_G(b)) + |V(G)|(\delta_H(c) + \delta_H(d))][\delta_G(a)\delta_H(c) + \delta_G(b)\delta_H(d)] \\
+ & [\delta_G(a)\delta_H(c) + \delta_G(b)\delta_H(d)]^2] \\
+ & \sum_{cd\in E(H)} \sum_{a\in V(G)} \sum_{b\in V(G)} [[|V(H)|(\delta_G(a) + \delta_G(b)) + |V(G)|(\delta_H(c) + \delta_H(d))]^2 \\
- & 2[|V(H)|(\delta_G(a) + \delta_G(b)) + |V(G)|(\delta_H(c) + \delta_H(d))][\delta_G(a)\delta_H(c) + \delta_G(b)\delta_H(d)] \\
+ & [\delta_G(a)\delta_H(c) + \delta_G(b)\delta_H(d)]^2] \\
- & 2\sum_{ab\in E(G)} \sum_{cd\in E(H)} [[|V(H)|(\delta_G(a) + \delta_G(b)) + |V(G)|(\delta_H(c) + \delta_H(d))]^2 \\
- & 2[|V(H)|(\delta_G(a) + \delta_G(b)) + |V(G)|(\delta_H(c) + \delta_H(d))][\delta_G(a)\delta_H(c) + \delta_G(b)\delta_H(d)] \\
+ & [\delta_G(a)\delta_H(c) + \delta_G(b)\delta_H(d)]^2],
\end{aligned}
$$

Step 1

$$\sum_{ab\in E(G)}\sum_{c\in V(H)}\sum_{d\in V(H)}[...]$$

$$\begin{aligned}
&= \sum_{ab\in E(G)}\sum_{c\in V(H)}\sum_{d\in V(H)}[|V(H)|^2(\delta_G(a)+\delta_G(b))^2+2|V(H)||V(G)|(\delta_G(a)+\delta_G(b))(\delta_H(c)+\delta_H(d))\\
&+ |V(G)|^2(\delta_H(c)+\delta_H(d))^2-2|V(H)|(\delta_G(a)+\delta_G(b))\delta_G(a)\delta_H(c)-2|V(H)|(\delta_G(a)+\delta_G(b))\delta_G(b)\delta_H(d)\\
&- 2|V(G)|(\delta_H(c)+\delta_H(d))\delta_G(a)\delta_H(c)-2|V(G)|(\delta_H(c)+\delta_H(d))\delta_G(b)\delta_H(d)+\delta_G^2(a)\delta_H^2(c)\\
&+ 2\delta_G(a)\delta_H(c)\delta_G(b)\delta_H(d)+\delta_G^2(b)\delta_H^2(d)],
\end{aligned}$$

$$\begin{aligned}
&= |V(H)|^2\sum_{ab\in E(G)}\sum_{c\in V(H)}\sum_{d\in V(H)}[\delta_G(a)+\delta_G(b)]^2\\
&+ 2|V(H)||V(G)|\sum_{ab\in E(G)}\sum_{c\in V(H)}\sum_{d\in V(H)}[\delta_G(a)+\delta_G(b))(\delta_H(c)+\delta_H(d)]\\
&+ |V(G)|^2\sum_{ab\in E(G)}\sum_{c\in V(H)}\sum_{d\in V(H)}[\delta_H(c)+\delta_H(d)]^2\\
&- 2|V(H)|\sum_{ab\in E(G)}\sum_{c\in V(H)}\sum_{d\in V(H)}[\delta_G(a)+\delta_G(b)]\delta_G(a)\delta_H(c)\\
&- 2|V(H)|\sum_{ab\in E(G)}\sum_{c\in V(H)}\sum_{d\in V(H)}[\delta_G(a)+\delta_G(b)]\delta_G(b)\delta_H(d)\\
&- 2|V(G)|\sum_{ab\in E(G)}\sum_{c\in V(H)}\sum_{d\in V(H)}[\delta_H(c)+\delta_H(d)]\delta_G(a)\delta_H(c)\\
&- 2|V(G)|\sum_{ab\in E(G)}\sum_{c\in V(H)}\sum_{d\in V(H)}[\delta_H(c)+\delta_H(d)]\delta_G(b)\delta_H(d)\\
&+ \sum_{ab\in E(G)}\sum_{c\in V(H)}\sum_{d\in V(H)}\delta_G^2(a)\delta_G^2(b)\\
&+ 2\sum_{ab\in E(G)}\sum_{c\in V(H)}\sum_{d\in V(H)}\delta_G(a)\delta_G(b)\delta_H(c)\delta_H(d)\\
&+ \sum_{ab\in E(G)}\sum_{c\in V(H)}\sum_{d\in V(H)}\delta_G^2(b)\delta_H^2(d),
\end{aligned}$$

$$\begin{aligned}
&= |V(H)|^4HM(G)+8|V(H)|^2|E(H)||V(G)|M_1(G)+|V(G)|^2|E(G)|[2|V(H)|M_1(H)+8|E(H)|]\\
&- 4|V(H)|^2|E(H)|\sum_{ab\in E(G)}[\delta_G^2(a)+\delta_G^2(b)]-4|V(H)|^2|E(H)|M_2(G)-4|V(H)|^2|E(H)|M_2(G)\\
&- 2|V(G)|[|V(H)|M_1(H)+4|E(H)|^2]\sum_{ab\in E(G)}[\delta_G(a)+\delta_G(b)]\\
&+ |V(H)|M_1(H)\sum_{ab\in E(G)}[\delta_G^2(a)+\delta_G^2(b)]+8|E(H)|M_2(G),
\end{aligned}$$

$$\begin{aligned}
&= |V(H)|^4HM(G)+8|V(H)|^2|E(H)||V(G)|M_1(G)+|V(G)|^2|E(G)|[2|V(H)|M_1(H)\\
&+ 8|E(H)|]-4|V(H)|^2|E(H)|F(G)-8|V(H)|^2|E(H)|M_2(G)-2|V(G)|M_1(G)[|V(H)|M_1(H)\\
&+ 4|E(H)|^2]+|V(H)|M_1(H)F(G)+8|E(H)|M_2(G).\quad\square
\end{aligned}$$

Step 2

$$\sum_{cd\in E(H)}\sum_{a\in V(G)}\sum_{b\in V(G)}[...]$$

$$\begin{aligned}
&= \sum_{cd\in E(H)}\sum_{a\in V(G)}\sum_{b\in V(G)}[|V(H)|^2(\delta_G(a)+\delta_G(b))^2+2|V(H)||V(G)|(\delta_G(a)+\delta_G(b))(\delta_H(c)+\delta_H(d))\\
&+ |V(G)|^2(\delta_H(c)+\delta_H(d))^2-2|V(H)|(\delta_G(a)+\delta_G(b))\delta_G(a)\delta_H(c)-2|V(H)|(\delta_G(a)+\delta_G(b))\delta_G(b)\delta_H(d)\\
&- 2|V(G)|(\delta_H(c)+\delta_H(d))\delta_G(a)\delta_H(c)-2|V(G)|(\delta_H(c)+\delta_H(d))\delta_G(b)\delta_H(d)+\delta_G^2(a)\delta_H^2(c)\\
&+ 2\delta_G(a)\delta_H(c)\delta_G(b)\delta_H(d)+\delta_G^2(b)\delta_H^2(d)],
\end{aligned}$$

$$\begin{aligned}
&= |V(H)|^2\sum_{cd\in E(H)}\sum_{a\in V(G)}\sum_{b\in V(G)}[\delta_G(a)+\delta_G(b)]^2\\
&+ 2|V(H)||V(G)|\sum_{cd\in E(H)}\sum_{a\in V(G)}\sum_{b\in V(G)}[\delta_G(a)+\delta_G(b))(\delta_H(c)+\delta_H(d)]\\
&+ |V(G)|^2\sum_{cd\in E(H)}\sum_{a\in V(G)}\sum_{b\in V(G)}[\delta_H(c)+\delta_H(d)]^2\\
&- 2|V(H)|\sum_{cd\in E(H)}\sum_{a\in V(G)}\sum_{b\in V(G)}[\delta_G(a)+\delta_G(b)]\delta_G(a)\delta_H(c)\\
&- 2|V(H)|\sum_{cd\in E(H)}\sum_{a\in V(G)}\sum_{b\in V(G)}[\delta_G(a)+\delta_G(b)]\delta_G(b)\delta_H(d)\\
&- 2|V(G)|\sum_{cd\in E(H)}\sum_{a\in V(G)}\sum_{b\in V(G)}[\delta_H(c)+\delta_H(d)]\delta_G(a)\delta_H(c)\\
&- 2|V(G)|\sum_{cd\in E(H)}\sum_{a\in V(G)}\sum_{b\in V(G)}[\delta_H(c)+\delta_H(d)]\delta_G(b)\delta_H(d)\\
&+ \sum_{cd\in E(H)}\sum_{a\in V(G)}\sum_{b\in V(G)}\delta_G^2(a)\delta_G^2(b)\\
&+ 2\sum_{cd\in E(H)}\sum_{a\in V(G)}\sum_{b\in V(G)}\delta_G(a)\delta_G(b)\delta_H(c)\delta_H(d)\\
&+ \sum_{cd\in E(H)}\sum_{a\in V(G)}\sum_{b\in V(G)}\delta_G^2(b)\delta_H^2(d),
\end{aligned}$$

$$\begin{aligned}
&= |V(H)|^2|E(H)|[2|V(G)|M_1(G)+8|E(G)|]+8|V(H)||V(G)|^2|E(G)|M_1(H)\\
&+ |V(G)|^4HM(H)-2|V(H)|[[|V(G)|M_1(G)+4|E(G)|^2]M_1(H)-4|V(G)|^2|E(G)|F(H)\\
&- 8|V(G)|^2|E(G)|M_2(H)+|V(G)|M_1(G)F(H)+8|E(G)|M_2(H). \quad \square
\end{aligned}$$

Step 3

$$-2\sum_{ab\in E(G)}\sum_{cd\in E(H)}[...]$$

$$\begin{aligned}
&= -2\sum_{ab\in E(G)}\sum_{cd\in E(H)}[|V(H)|^2(\delta_G(a)+\delta_G(b))^2+2|V(H)||V(G)|(\delta_G(a)+\delta_G(b))(\delta_H(c)+\delta_H(d))\\
&+ |V(G)|^2(\delta_H(c)+\delta_H(d))^2-2|V(H)|(\delta_G(a)+\delta_G(b))\delta_G(a)\delta_H(c)-2|V(H)|(\delta_G(a)+\delta_G(b))\delta_G(b)\delta_H(d)\\
&- 2|V(G)|(\delta_H(c)+\delta_H(d))\delta_G(a)\delta_H(c)-2|V(G)|(\delta_H(c)+\delta_H(d))\delta_G(b)\delta_H(d)+\delta_G^2(a)\delta_H^2(c)\\
&+ 2\delta_G(a)\delta_H(c)\delta_G(b)\delta_H(d)+\delta_G^2(b)\delta_H^2(d)],
\end{aligned}$$

$$\begin{aligned}
&= -2|V(H)|^2 \sum_{ab\in E(G)} \sum_{cd\in E(H)} [\delta_G(a)+\delta_G(b)]^2 \\
&- 4|V(H)||V(G)| \sum_{ab\in E(G)} \sum_{cd\in E(H)} [\delta_G(a)+\delta_G(b))(\delta_H(c)+\delta_H(d)] \\
&- 2|V(G)|^2 \sum_{ab\in E(G)} \sum_{cd\in E(H)} [\delta_H(c)+\delta_H(d)]^2 \\
&+ 4|V(H) \sum_{ab\in E(G)} \sum_{cd\in E(H)} [\delta_G(a)+\delta_G(b)]\delta_G(a)\delta_H(c) \\
&+ 4|V(H)| \sum_{ab\in E(G)} \sum_{cd\in E(H)} [\delta_G(a)+\delta_G(b)]\delta_G(b)\delta_H(d) \\
&+ 4|V(G) \sum_{ab\in E(G)} \sum_{cd\in E(H)} [\delta_H(c)+\delta_H(d)]\delta_G(a)\delta_H(c) \\
&+ 4|V(G) \sum_{ab\in E(G)} \sum_{cd\in E(H)} [\delta_H(c)+\delta_H(d)]\delta_G(b)\delta_H(d) \\
&- 2 \sum_{ab\in E(G)} \sum_{cd\in E(H)} \delta_G^2(a)\delta_G^2(b) \\
&- 4 \sum_{ab\in E(G)} \sum_{cd\in E(H)} \delta_G(a)\delta_G(b)\delta_H(c)\delta_H(d) \\
&- 2 \sum_{ab\in E(G)} \sum_{cd\in E(H)} \delta_G^2(b)\delta_H^2(d),
\end{aligned}$$

$$\begin{aligned}
&= -2|V(H)|^2|E(H)|HM(G) - 4|V(H)||V(G)|M_1(G)M_1(H) - 2|V(G)|^2|E(G)|HM(H) \\
&+ 4|V(H)|M_1(H)F(G) + 4|V(H)|M_2(G)M_1(H) + 4|V(G)|M_1(G)F(H) \\
&+ 4|V(G)|M_2(H)M_1(G) - 2F(G)F(H) - 4M_2(G)M_2(H). \quad \square
\end{aligned}$$

By summation of step1, step2 and step3 complete the proof.

The following theorem gives the Hyper-zagreb index of the tensor product of graphs.

Theorem 3.2: Let G and H be graphs. Then:

$$HM(G \otimes H) = F(G)F(H) + 2M_2(G)M_2(H).$$

Proof. By Definition 2.1 and Lemma 2.2, we have

$$HM(G \otimes H)$$

$$\begin{aligned}
&= \sum_{(a,c)(b,d)\in E(G\otimes H)} [\delta_{G\otimes H}(a,c)+\delta_{G\otimes H}(b,d)]^2 \\
&= \sum_{(a,c)(b,d)\in E(G\otimes H)} [\delta_G(a)\delta_H(c)+\delta_G(b)\delta_H(d)]^2 \\
&= \sum_{ab\in E(G)}\sum_{cd\in E(H)} [\delta_G^2(a)\delta_H^2(c)+2\delta_G(a)\delta_H(c)\delta_G(b)\delta_H(d)+\delta_G^2(b)\delta_H^2(d)] \\
&= \sum_{ab\in E(G)}\sum_{cd\in E(H)} [\delta_G^2(a)\delta_H^2(c)+\delta_G^2(b)\delta_H^2(d)] \\
&+ 2\sum_{ab\in E(G)}\sum_{cd\in E(H)} [\delta_G(a)\delta_H(c)\delta_G(b)\delta_H(d)] \\
&= F(G)F(H)+2M_2(G)M_2(H), \quad \square
\end{aligned}$$

The following theorem gives the Hyper-zagreb index of the symmetric difference $G \oplus H$ of two graphs G and H.

Theorem 3.3: Let G and H be graphs. Then:

$HM(G \oplus H) =$

$[|V(G)|^4 - 4|V(H)|^2|E(H)|]HM(H) + [|V(H)|^4 - 4|V(H)|^2|E(H)|]HM(G) + 20|V(G)|M_1(G)F(H) + 20|V(H)|M_1(H)F(G) + 10|V(H)|^2|E(H)||V(G)|M_1(G) + 10|V(H)||V(G)|^2|E(G)|M_1(H) + 8|V(H)|^2|E(H)||E(G)| + -16|V(H)||E(G)|^2M_1(H) - 16|V(G)||E(H)|^2M_1(G) - 8|V(G)|^2|E(G)|F(H) - 8|V(H)|^2|E(H)|F(G) - 16|V(G)|^2|E(G)|M_2(H) - 16|V(H)|^2|E(H)|M_2(G) + 32|E(G)|M_2(H) + 32|E(H)|M_2(G) - 16|V(H)||V(G)|M_1(G)M_1(H) + 16|V(H)|M_2(G)M_1(H) + 16|V(G)|M_2(H)M_1(G) - 16F(G)F(H) - 32M_2(G)M_2(H)$.

Proof. By Definition 2.1 and Lemma 2.2, and by the same method as Theorem 3.1 we have:

$HM(G \oplus H) =$

$$\begin{aligned}
&= \sum_{(a,c)(b,d)\in E(G\oplus H)} [\delta_{(G\oplus H)}(a,c)+\delta_{(G\oplus H)}(b,d)]^2 \\
&= \sum_{(a,c)(b,d)\in E(G\oplus H)} [|V(H)|\delta_G(a)+|V(G)|\delta_H(c)-2\delta_G(a)\delta_H(c)+|V(H)|\delta_G(b) \\
&+ |V(G)|\delta_H(d)-2\delta_G(b)\delta_H(d)]^2 \\
&= \sum_{ab\in E(G)}\sum_{c\in V(H)}\sum_{d\in V(H)} [|V(H)|\delta_G(a)+|V(G)|\delta_H(c)-2\delta_G(a)\delta_H(c)+|V(H)|\delta_G(b) \\
&+ |V(G)|\delta_H(d)-2\delta_G(b)\delta_H(d)]^2 \\
&+ \sum_{cd\in E(H)}\sum_{a\in V(G)}\sum_{b\in V(G)} [|V(H)|\delta_G(a)+|V(G)|\delta_H(c)-2\delta_G(a)\delta_H(c)+|V(H)|\delta_G(b) \\
&+ |V(G)|\delta_H(d)-2\delta_G(b)\delta_H(d)]^2 \\
&- 4\sum_{ab\in E(G)}\sum_{cd\in E(H)} [|V(H)|\delta_G(a)+|V(G)|\delta_H(c)-2\delta_G(a)\delta_H(c)+|V(H)|\delta_G(b) \\
&+ |V(G)|\delta_H(d)-2\delta_G(b)\delta_H(d)]^2,
\end{aligned}$$

Step 1

$$\sum_{ab\in E(G)}\sum_{c\in V(H)}\sum_{d\in V(H)}[...]$$

$$\begin{aligned}
&= |V(H)|^4HM(G)+8|V(H)|^2|E(H)||V(G)|M_1(G)+|V(G)|^2|E(G)|[2|V(H)|M_1(H)\\
&+ 8|E(H)|]-8|V(H)|^2|E(H)|F(G)-16|V(H)|^2|E(H)|M_2(G)-4|V(G)|M_1(G)[|V(H)|M_1(H)\\
&+ 4|E(H)|^2]+4|V(H)|M_1(H)F(G)+32|E(H)|M_2(G) \quad \square
\end{aligned}$$

Step 2

$$\sum_{cd\in E(H)}\sum_{a\in V(G)}\sum_{b\in V(G)}[...]$$

$$\begin{aligned}
&= |V(H)|^2|E(H)|[2|V(G)|M_1(G)+8|E(G)|]+8|V(H)||V(G)|^2|E(G)|M_1(H)\\
&+ |V(G)|^4HM(H)-4|V(H)|[|V(G)|M_1(G)+4|E(G)|^2]M_1(H)-8|V(G)|^2|E(G)|]F(H)\\
&- 16|V(G)|^2|E(G)|M_2(H)+4|V(G)|M_1(G)F(H)+32|E(G)|M_2(H) \quad \square
\end{aligned}$$

Step 3

$$-4\sum_{ab\in E(G)}\sum_{cd\in E(H)}[...]$$

$$\begin{aligned}
&= -4|V(H)|^2|E(H)|HM(G)-8|V(H)||V(G)|M_1(G)M_1(H)-4|V(G)|^2|E(G)|HM(H)\\
&+ 16|V(H)|M_1(H)F(G)+16|V(H)|M_2(G)M_1(H)+16|V(G)|M_1(G)F(H)\\
&+ 16|V(G)|M_2(H)M_1(G)-16F(G)F(H)-32M_2(G)M_2(H) \quad \square
\end{aligned}$$

By summation of step1, step2 and step3 complete the proof.

Corollary 3.4: Let $T = T[p,q]$ be the molecular graph of a nanotorus such that $|V(T)| = pq$, $|E(T)| = \{3/2\}pq$, which given in Figure 2. Then,

$$HM(T) = \sum_{uv\in E(T)} (\delta_T(u) + \delta_T(v))^2$$

$$= \sum_{uv\in E(T)} \left(\delta_T^2(u) + 2\delta_T(u)\delta_T(v) + \delta_T^2(v)\right)$$

$$= F(T) + 2M_2(T)$$

By [13, 5] $M_2(T) = 27/2pq$ and $F(T) = 27pq$, thus: $HM(T) = 54pq$. Also,

$$HM(P_n \times T) = |V(T)|HM(P_n) + |V(P_n)|HM(T) + 12|E(P_n)|M_1(T) + 12|E(T)|M_1(P_n)$$

$$= pq(16n-30) + 54pqn + 12(n-1)9pq + 123/2pq(4n-6)$$

$$= 250npq - 186pq$$

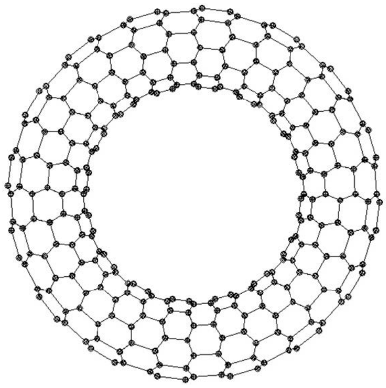

Fig. 2. The molecular graph of a nanotorus

4 Remarks

Remark 4.1: Let G and H be graphs with$m, n, k \geq 3$.Then:

1. $HM(P_n+C_m) = 16n-30+2m(20n-7)+8m[n^2+(m+1)(n-1)]+nm[(m+n)^2+4((n-1)+m)]$,
2. $HM(P_n \circ C_m) = m^4(16n-30)+64nm+12m^3n-32m$,
3. $HM(P_n * C_m) = nm^4+2m^2[7n-4]+76nm+16n-42m-30$,
4. $HM(P_n \otimes C_m) = 96mn-176m$,
5. $HM(P_n \vee C_m) = [n^4-2m^3]16m+[m^4-2m^3](16n-30)+10m^3n(4n-6)+40mn^2(n-1)m+8m^3(n-1)+8n^2(n-1)m-32m^2(n-1)^2-8nm^2(4n-6)-64n^2(n-1)m-4m^3(8n-14)-32m^3(n-2)++64(n-2)m^(2)+224n^2m+160m^2n-280m^2-464mn+256m$.

Remark 4.2: Let G and H be graphs with$m, n, k \geq 3$.Then:

1. $HM(C_k+C_m) = 16k+16m+5(4km+4mk)+8[k^2m+m^2k+km]+km[(m+k)^2+4(k+m)]$,
2. $HM(C_k \circ C_m) = 16m^4k+48m^3k+40m^2k+24km$,
3. $HM(C_k * C_m) = km^4+14km^2+76km+16k$,
4. $HM(C_k \otimes C_m) = 96km$,
5. $HM(P_n \vee C_m) = 16k^4m-32m^4+16m^4k-88m^3k+40m^3k^2+40m^2k^3-56k^3m-64m^2k^2+224m^2k+224k^2m-128km$.

Proof. For the proof (Remark 4.1 and Remark 4.2) we refer to Theorem 2.6, Theorem 3.1, Theorem 3.2 and Theorem 3.3

Remark 4.3: We investigated the correlation between the topological indices by statistical parameters as correlation coefficient for example the correlation matrix reflecting the linear correlation between the three indices (first Zagreb index, second Zagreb index, Hyper-Zagreb index) were computed for C_3 to C_{14}alkanes and cyclic alkanes shown in Table 1.

Table 1. Topological Indices for C_3 to C_{14} alkanes,and cyclic alkanes

n	$M_1(P_n)$	$M_2(P_n)$	$HM(P_n)$		m	$M_1(C_m)$	$M_2(C_m)$	$HM(C_m)$	
3	6	4	18		3	12	12	48	
4	10	8	34		4	16	16	64	
5	14	12	50		5	20	20	80	
6	18	16	66		6	24	24	96	
7	22	20	82		7	28	28	112	
8	26	24	98		8	32	32	128	
9	30	28	114		9	36	36	144	
10	34	32	130		10	40	40	160	
11	38	36	146		11	44	44	176	
12	42	40	162		12	48	48	192	
13	46	44	178		13	52	52	208	
14	50	48	194		14	56	56	224	

5 Conclusion

Hyper-Zagreb indices are a pair of recently introduced graph invariants that generalize much used Zagreb indices. In this paper, we have investigated some of the basic mathematical properties of the Hyper-Zagreb index and obtained explicit formula for their values under some graph binary operations such disjunction $G \vee H$, symmetric difference $G \oplus H$, and tensor product $G \otimes H$ of graphs. Much work still needs to be done, and here we mention some possible directions for future research as multiplicative Hyper-Zagreb indices and co-indices.

Competing Interests

Authors have declared that no competing interests exist.

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