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Some Characterizations of Whole Edge Domination in Bipolar Fuzzy Graphs

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, Some bounds, theorems and results on whole edge domination number in bipolar fuzzy graph are established with support of some examples. The concepts of perfect, complete perfect and semi-perfect whole edge domination in bipolar fuzzy graph are discussed and investigated with some of their properties and also results on perfect contributed via the support of some examples.

Keywords: Bipolar fuzzy graph (BFG); strong edge; Domination number; Edge domination number; Whole edge domination number.

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1 Introduction

Graph theory is now a new language that covers all the disciplines, including the literary sciences. Through its straightforward methods, it can provide an alternative perspective on most scientific issues. Domination is one of the most significant issues that graph theory addresses because it has numerous applications across most disciplines. By Mitchell and Hedetniemi [1], the idea of edge domination in graphs was first established. In 1965, Zadeh [2] developed the idea of a fuzzy subset of a set as a means of expressing uncertainty. Some types of domination in fuzzy graphs have been researched recently. Most of them are part of fuzzy graphs' vertex domination[3]. The researcher's motivation for investigating edge domination in fuzzy graphs.

In 1994, Zhang [4, 5] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. Kauffman in the year 1973, introduced the basic idea of fuzzy graph. After two years, the concept of fuzzy graphs was established by Rosenfeld [6]. Further, in [7] The extension of fuzzy graph into bipolar fuzzy graph was done by Akram. Karunambigai et al. in [8], defined the domination, the domination number in bipolar fuzzy graphs.

Fuzzy set extensions with a membership degree range of [-1,1] are known as bipolar fuzzy sets. In a bipolar fuzzy set, an element's membership degree of 0 indicates that it has no impact on the corresponding property, its membership degree of [0,1] that it somewhat satisfies the property, and its membership degree of [-1,0] that it somewhat satisfies the implicit counter-property. In [9], the domination will be calculated by means of edge sets.

In this paper, Section 2 deals with basic definions related to this topic. Many bounds and properties of whole edge domination in bipolar fuzzy graph have been determined in section 3. Moreover, for certain graphs, this number has been introduced. In section 4, the effect of addition,deletion and contraction of an edge on the perfect whole domination on BFG has been calculated. Finally, the section gives the conclusion of the article.

2 Preliminaries

Definition 2.1. [7] A fuzzy subset μ on a set X is a map $\mu : X \to [0,1]$. A map $v : X \times X \to [0,1]$ is called a fuzzy relation on X if $v(x,y) \leq min(\mu(x), \mu(y))$ for all $x, y \in X$.

Definition 2.2. [7] Let X be a non empty set. A bipolar fuzzy set M in X is an object having the form $B = \{(x, \mu_B^+, \mu_B^-)/x \in X\}$ where, $\mu_B^+ : X \to [0,1]$ and $\mu_B^- : X \to [-1,0]$ are mappings.

Definition 2.3. [10] A Bipolar fuzzy graph (BFG) is of the form $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where

1. $V = v_1, v_2, v_3, ...v_n$ such that $\mu_1^+ : X \to [0,1]$ and $\mu_1^- : X \to [-1,0]$.

2. $\varepsilon \subset V \times V$ where $\mu_2^+ : V \times V \to [0,1]$ and $\mu_2^- : V \times V \to [-1,0]$ such that
   $\mu_{2ij}^+ = \mu_2^+(v_i, v_j) \leq min(\mu_1^+(v_i), \mu_1^+(v_j))$ and
   $\mu_{2ij}^- = \mu_2^-(v_i, v_j) \geq max(\mu_1^-(v_i), \mu_1^-(v_j))$ for all $(v_i, v_j) \in E$.

Definition 2.4. [10] Let u be a vertex in a BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ then $N(u) = \{v : v \in V\}$ and (u, v) is a strong edge in $\mathcal{G}$ is called neighbourhood of u in $\mathcal{G}$.

Definition 2.5. [11] A set $\mathcal{D}_E \subseteq \mathcal{E}$ is said to be an edge dominating set if every edge in $\mathcal{E} - \mathcal{D}_E$ is adjacent to some edge in $\mathcal{D}_E$. The edge domination number of $\mathcal{G}$ is the cardinality of a smallest edge dominating set of $\mathcal{G}$ and is denoted by γ.

Definition 2.6. [9] In a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a proper subset $\mathcal{D}_W \subseteq E$ is called whole edge dominating set ($WEDS$), if every edge in $\mathcal{D}_W$ is adjacent to all edges in $\mathcal{E} - \mathcal{D}_W$.

Definition 2.7. [9] In a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, If X is a $WEDS$, then $\mathcal{D}_W$ is called minimal $WEDS$, if it has no proper $WEDS$.

Definition 2.8. [9] A minimal $WEDS$ has smallest cardinality is called whole edge domination number denoted by $\gamma_{whe}(G)$.

Definition 2.9. [11] The number of edges (the cardinality of E) is called the size of a bipolar fuzzy graph (BFG) and is denoted by
$S(\mathcal{G}) = \sum_{v_i, v_j \in \mathcal{E}} \left(\dfrac{1 + \mu_2^+(v_i, v_j) + \mu_2^-(v_i, v_j)}{2} \right)$, for all $(v_i, vj) \in \mathcal{E}$.

Definition 2.10. [11] In BFG, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, an edge (a, b) is said to be strong edge if
$\mu_2^+(a, b) \geq (\mu_2^+)^\infty(a, b)$ and $\mu_2^-(a, b) \leq (\mu_2^-)^\infty(a, b)$ where

$$(\mu_2^+)^\infty(a, b) = max\{(\mu_2^+)^k(a, b)/k = 1, 2, 3, ...n\} \text{ and}$$
$$(\mu_2^-)^\infty(a, b) = min\{(\mu_2^-)^k(a, b)/k = 1, 2, 3, ...n\}.$$

Definition 2.11. [11] Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an BFG. Let e_i and e_j be two edges of $\mathcal{G}$. We say that e_i dominates e_j, if e_i is a strong arc in $\mathcal{G}$ and adjacent to e_j.

Definition 2.12. [7] A BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called strong if
$\mu_2^+(xy) = min\{\mu_1^+(x), \mu_1^+(y)\}$ and $\mu_2^-(xy) = min\{\mu_1^-(x), \mu_1^-(y)\}$ for all $xy \in \mathcal{E}$.

Definition 2.13. [7] The complement of a strong BFG $\mathcal{G} = (A, B)$ of $\mathcal{G}^* = (\mathcal{V}, \mathcal{E})$ is a strong BFG $\bar{\mathcal{G}} = (\bar{A}, \bar{B})$ on $\bar{\mathcal{G}}$, where $\bar{A} = (\bar{\mu}_A^+, \bar{\mu}_A^-)$ and $\bar{B} = (\bar{\mu}_B^+, \bar{\mu}_B^-)$ are defined by

- $\bar{\mathcal{V}} = \mathcal{V}$
- $\bar{\mu}_A^+ = \mu_A^+$ and $\bar{\mu}_A^- = \mu_A^-$, for all $x \in \mathcal{V}$
-

$$\bar{\mu_B^+}(xy) = \begin{cases} 0 & if \mu_B^+(xy) > 0 \\ min(\mu_A^+(x), \mu_A^+(y)) & if \mu_B^+(xy) = 0 \end{cases}$$

$$\bar{\mu_B}(xy) = \begin{cases} 0 & if \mu_B^-(xy) < 0 \\ min(\mu_A^-(x), \mu_A^-(y)) & if \mu_B^-(xy) = 0 \end{cases}$$

Definition 2.14. [7] A strong BFG $\mathcal{G}$ is called self complementary if $\bar{\bar{\mathcal{G}}} = \mathcal{G}$.

3 Whole Edge Domination in Bipolar Fuzzy Graph

Definition 3.1. In a BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a proper subset $\mathcal{D}_W \subseteq \mathcal{E}$ is called whole edge dominating set ($WEDS$), if every edge in $\mathcal{D}_W$ is strong to all edges in $\mathcal{E} - \mathcal{D}_W$.

Definition 3.2. In a BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, If $\mathcal{D}_W$ is a $WEDS$, then $\mathcal{D}_W$ is called minimal $WEDS$, if it has no proper $WEDS$.

Definition 3.3. A minimal $WEDS$ in BFG has smallest cardinality is called whole edge domination number in BFG denoted by $\gamma_{whe}(\mathcal{G})$.

Definition 3.4. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a BFG and $e \in \mathcal{E}$, when we delete an edge e from $\mathcal{G}$ then the edges of $\mathcal{G}$ are partition into two sets

$$\mathcal{E}_{-}^{0} = \{e \in \mathcal{E}, \gamma_{whe}(\mathcal{G} - e) = \gamma(\mathcal{G})\},$$
$$\mathcal{E}_{-}^{-} = \{e \in \mathcal{E}, \gamma_{whe}(\mathcal{G} - e) < \gamma(\mathcal{G})\}.$$

Definition 3.5. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a BFG and $e \in \mathcal{G}$, when we add an edge e from $\mathcal{G}$ then the edges of $\mathcal{G}$ are partition into two sets

$$\mathcal{E}_{+}^{0} = \{e \in \mathcal{E}, \gamma_{whe}(\mathcal{G} + e) = \gamma(\mathcal{G})\},$$
$$\mathcal{E}_{+}^{-} = \{e \in \mathcal{E}, \gamma_{whe}(\mathcal{G} + e) < \gamma(\mathcal{G})\}.$$

Example 3.1. *Consider a BFG, (Fig. 1:),*

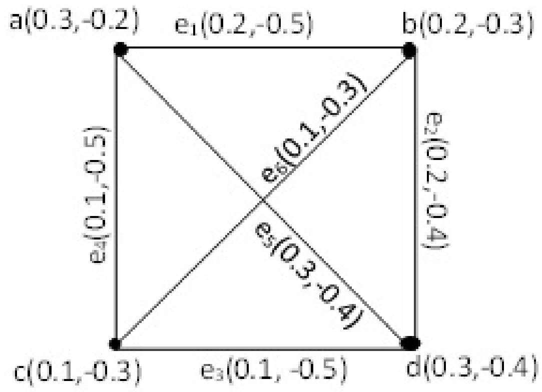

Fig. 1. Edge Deletion in BFG

If we remove the edge e_3, then $\gamma_{whe}(\mathcal{G} - e) < \gamma(\mathcal{G})$.

Example 3.2. *Consider a BFG, (Fig. 2:),*

If we add the edge e (Fig. 2), then $\gamma_{whe}(\mathcal{G} + e) < \gamma(\mathcal{G})$.

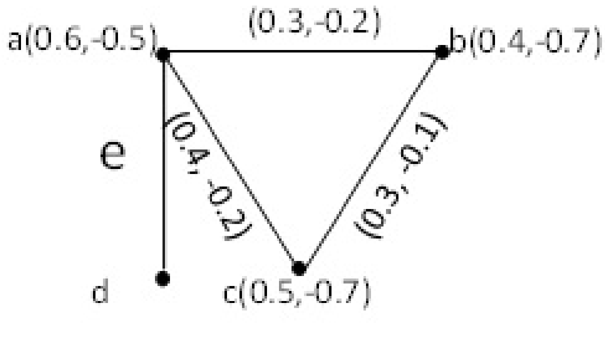

Fig. 2. Edge Addition in BFG

Example 3.3. *Consider a BFG, (Fig. 3:),* *If we add the edge e,(Fig. 3), then* $\gamma_{whe}(\mathcal{G}+e)=\gamma(\mathcal{G})$.

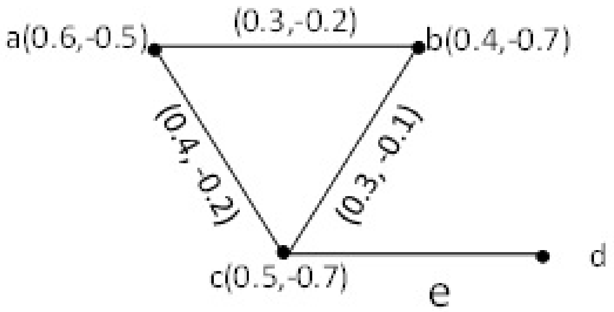

Fig. 3. Edge Addition in BFG(1)

Theorem 3.4. *If a BFG* $\mathcal{G}=(\mathcal{V},\mathcal{E})$ *has a whole edge domination number* $\gamma_{whe}(\mathcal{G})$, *then* $\mathcal{E}_{+}^{0},\mathcal{E}_{+}^{-}$ *and* $\mathcal{E}_{+}^{+}$ *are not empty sets.*

Proof. Two cases are appear as follows.
Case :(i)
If a graph $\mathcal{G}$ is a star and add an edge incident to the vertices then the BFG $(\mathcal{G}+e)$ also has $\gamma_{whe}(\mathcal{G}+e)$.
Case :(ii)
If there is no edge in $\mathcal{D}_W$ is adjacent to the addition an edge e, then two cases are appear as follows.

i) If we take $\mathcal{G}=P_4$ and add the edge that is incident to the two pendants vertices of P_4, so the BFG $\mathcal{G}+e=C_4$, therefore, the BFG $\mathcal{G}+e$ also has $\gamma_{whe}(\mathcal{G}+e)$.
ii) If we take $C(\mathcal{G})=C_4$ and add an edge incident to the vertices, so $\mathcal{G}+e$ contains an edge adjacent to all edges. Then the BFG $\mathcal{G}+e$ also has $\gamma_{whe}(\mathcal{G}+e)$. □

Theorem 3.5. *If a BFG* $\mathcal{G}=(\mathcal{V},\mathcal{E})$ *has a whole edge domination number* γ_{whe}, *then* $\mathcal{E}_{-}^{0},\mathcal{E}_{-}^{-}$ *and* $\mathcal{E}_{-}^{+}$ *are not empty sets.*

Proof. As same manner in the previous theorem by deleting the edge. □

Corollary 3.6.

- *For the path BFG* P_n *with* $n\geq 3$,
  1. *If* $3\leq n\leq 4$ *then the BFG has* $\gamma_{whe}(P_n)$.
  2. *If* $n\geq 5$ *then the BFG has no* $\gamma_{whe}(P_n)$.

- *For the cycle BFG* C_n *with* $n\geq 3$,
  1. *If* $3\leq n\leq 4$ *then the BFG has* $\gamma_{whe}(C_n)$.
  2. *If* $n\geq 5$ *then the BFG has no* $\gamma_{whe}(C_n)$.

- *For the complete BFG* K_n *with* $n\geq 3$,
  1. *If* $3\leq n\leq 4$ *then the BFG has* $\gamma_{whe}(K_n)$.
  2. *If* $n\geq 5$ *then the BFG has no* $\gamma_{whe}(K_n)$.

- *For a Wheel BFG* W_n *with* $n\geq 3$,
  1. *If* $n=3$, *then the BFG has* $\gamma_{whe}(W_n)$.
  2. *If* $n\geq 4$ *then the BFG has no* $\gamma_{whe}(W_n)$.

- *If BFG be a star S_n with $n \geq 3$,*
  then the star has $\gamma_{whe}(S_n)$.

- *If BFG be a double star $S_{m,n}$, then the double star has $\gamma_{whe}(S_{m,n})$.*

Theorem 3.7. *A BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a tree (T) and it has one whole edge domination number if the number of vertices not more than 3.*

Proof. Let T be a tree and let $\mathcal{D}_W$ be a whole edge dominating set with minimum cardinality.
Suppose that $\mathcal{D}_W$ contains two edges say $\{e_1, e_2\}$, then there are two cases arises.

Case:(i)
If the number of the remained edges in $\mathcal{G} - \mathcal{D}_W$ is one say $\{e_3\}$, then since the graph is a tree so it must be a path of some order with an edge $\{e_3\}$ which the incident vertices on it of some degree.
Thus, $\{e_3\}$ is a whole edge dominating set in BFG and this is a contradiction with the set $\mathcal{D}_W$ is the minimum cardinality.

Case:(ii)
If the number of the remained edges in $\mathcal{G} - \mathcal{D}_W$ more than one, then there is a cycle contains the edges in $\mathcal{D}_W$ and the other edges.

Again, this is a contradiction with our assumption.

Therefore, $\mathcal{D}_W$ has one edge, then the middle edge in this tree is whole edge dominating set, which is strong to all edges.
So, that edge has minimum number of whole edge dominating graph.

Thus, the required is satisfied. □

Theorem 3.8. *Let $\mathcal{G}$ be BFG and $\bar{\mathcal{G}}$ be the complement of BFG $\mathcal{G}$ with the nodes and arcs as in G or not. If $\mathcal{D}_W$ is the whole edge dominating set of $\mathcal{G}$ then $\bar{\mathcal{G}}$ also has atleast one whole edge dominating set.*

Proof. Let $\mathcal{G}$ and $\bar{\mathcal{G}}$ be BFG . Let us assume that $\mathcal{G}$ contains less number of nodes and arcs than $\mathcal{G}$ or equal number of nodes and arcs of $\mathcal{G}$.

Suppose e_i and e_j are any two edges adjacent in $\mathcal{G}$ then they may be adjacent (or) non adjacent in $\bar{\mathcal{G}}$.

This implies there exists distinct edge dominating sets in $\bar{\mathcal{G}}$ but which does not equals $\mathcal{D}_W$. □

Theorem 3.9. *If $\mathcal{D}_W$ be whole edge dominating set of a complete BFG $\mathcal{G}$, then the edges of whole edge dominating set $\mathcal{D}_W$ incident with the nodes containing maximum degree.*

Proof. Let $\mathcal{D}_W$ be a whole edge dominating set in $\mathcal{G}$.

Assume that the edges of whole edge dominating set $\mathcal{D}_W$ is not incident with the nodes having maximum degree. Then arcs of whole edge dominating set $\mathcal{D}_W$ are strong, which are incident with the node containing minimum degree.

By definition of edge dominating set, for each $e_j \in \mathcal{E} - \mathcal{D}_W$ there exists $e_i \in \mathcal{D}_W$ such that e_i is strong to e_j. Hence whole edge dominating set $\mathcal{D}_W$ must contain atleast one strong arc.

This implies $\mathcal{D}_W$ is not minimum, then it leads to contradiction.

Hence edges of whole edge dominating set $\mathcal{D}_W$ should incident with the nodes containing maximum degree. □

Theorem 3.10. *Consider a BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, then $\mathcal{G}$ has $\gamma_{whe}(\mathcal{G})$ if $\gamma_{whe}(\mathcal{G}/e)$.*

Proof. **Case:(i)**
a) If $\mathcal{G}$ contains a spanning subgraph isomorphic to star. This graph becomes a star too when we contract the edge e. So, the remains BFG $\mathcal{G}$ also has $\gamma_{whe}(\mathcal{G})$.
b) If BFG $\mathcal{G}$ be a double star, then we contract the edge e of the BFG $\mathcal{G}$ which is belong to the $\gamma_{whe}(\mathcal{G})$-set and that edge e separate the BFG $\mathcal{G}$ into the two stars. Then the BFG $\mathcal{G}$ is not connected.So it has no $\gamma_{whe}(\mathcal{G})$.
c) If the BFG $\mathcal{G}$ be a path, then we contract the edge e of the BFG $\mathcal{G}$ which is belong to the $\gamma_{whe}(\mathcal{G})$-set then the BFG $\mathcal{G}$ becomes not connected. So,the BFG $\mathcal{G}$ has no $\gamma_{whe}(\mathcal{G})$.
d) If the BFG $\mathcal{G}$ be a cycle, For example C_4, Then when we contract the edge e graph $\mathcal{G}$ turn to C_3, which means it has the whole edge domination number.

Case :(ii)
If e does not belong to any $\gamma_{whe}()G$-set, then contracting an edge e do not influence to whole domination number of $\mathcal{G}$. □

Remark:
1. If BFG $\mathcal{G} - v$ has whole edge domination number, then BFG $\mathcal{G}$ is not necessary has whole edge domination number.
2. If $\mathcal{G} - e$ has a whole edge domination number, then $\mathcal{G}$ is not necessary has whole edge domination number (as an example see Fig. 4).

Example 3.11. *Consider a BFG, (Fig. 4:),*

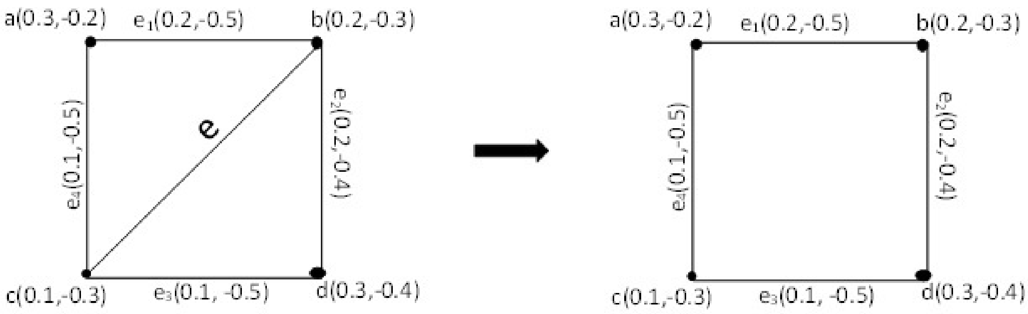

Fig. 4. Edge Removal in BFG (1)

In this example, If we remove the edge e, then $\gamma_{whe}(\mathcal{G} - e) < \gamma(\mathcal{G})$.

3. If $\mathcal{G} + e$ has whole edge domination number, then $\mathcal{G}$ not necessary has whole edge domination number (as an example see Fig. 5).

Example 3.12. *Consider a BFG, (Fig. 5:),*

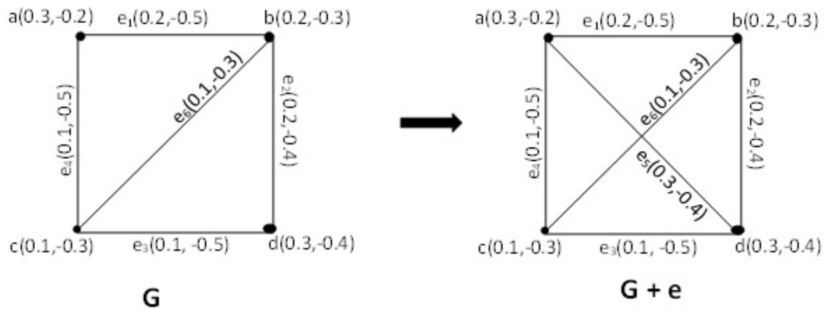

Fig. 5. Edge Addition in BFG (2)

In this example, If we add an edge e_5*, then* $\gamma_{whe}(\mathcal{G}+e) < \gamma(\mathcal{G})$.

4. If $\mathcal{G}+e$ has a whole edge domination number, then $\mathcal{G}$ is not necessary has a whole edge domination number.

Theorem 3.13. *If a BFG* $\mathcal{G}$ *has* γ_{whe}*, then* $\gamma_{whe}(\mathcal{G}-v) = \gamma_{whe}(\mathcal{G})$*,where* $v \in \mathcal{V}$ *(or)* $(\mathcal{G}-v)$ *has no whole edge domination set.*

Proof. There are two cases as follows.

Case :(i)
If $(\mathcal{G}-v)$ is disconnected, then $(\mathcal{G}-v)$ has no whole dominating set.
Case :(ii)
If $(\mathcal{G}-v)$ is connected,there are two cases as follows
i) If $\gamma_{whe}(\mathcal{G}) = 1$, then BFG $\mathcal{G}$ includes a spanning subgraph either it is a star or double star.
Now, If a BFG $\mathcal{G}$ includes a spanning subgraph isometric to star, then there are two cases as follows.

a) If a BFG $\mathcal{G}$ has maximum number of edges, which means there is an edge say $e = vu$, such that u and v are adjacent to all other vertices.

Thus, if we delete any other vertex from this graph the whole edge dominating is not influenced by this deletion, that means a BFG $\mathcal{G}$ has $\gamma_{whe}(\mathcal{G})$ (as an example, see Fig. 6).

Example 3.14. *Consider a BFG, (Fig. 6:),*

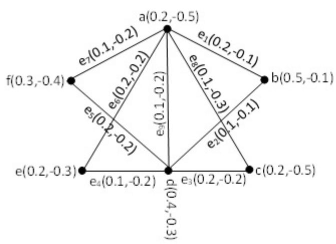

Fig. 6. Edge Removal in BFG (2)

In this example, If we remove the edge e_3*, then* $\gamma_{whe}(\mathcal{G}-e) < \gamma(\mathcal{G})$.

b) If a BFG $\mathcal{G}$ has no maximum number of strong edges, which means the vertex u that is incident with the edge e is not adjacent to some other vertices in $\mathcal{G}$, so if we delete the vertex v, then we get an isolated vertex,

so $(G - v)$ has no whole edge dominating set.

Otherwise, deleting any vertex from graph $\mathcal{G}$ do not influence the whole edge domination.

Now, if BFG $\mathcal{G}$ contains a spanning subgraph isometric to double star, then there are two cases as follows.
c) If $e = vu$ is the edge that is strong from all the edges in $\mathcal{G}$, and if we delete u or v, then the graph $(\mathcal{G} - v)$ or $(\mathcal{G} - u)$ has an isolated vertices. Thus $(\mathcal{G} - v)$ or $(\mathcal{G} - u)$ has no whole edge dominating set.
d) If the deleted vertex is not adjacent to the edge e which is dominating the graph edges, then this deletion do not influence to whole edge domination edge.
If a BFG $\mathcal{G}$, it has strong edges $\mathcal{G} = C_4$ or K_4.

Thus, $(\mathcal{G} - v)$ is a path of order three, so it has $\gamma_{whe}(\mathcal{G} - v)$.

Therefore, $\gamma_{whe}(\mathcal{G} - v) = \gamma_{whe}(\mathcal{G})$.
For all cases above, one can see that $\gamma_{whe}(\mathcal{G} - v) = \gamma_{whe}(\mathcal{G})$ or $(\mathcal{G} - v)$ has no whole edge dominating set. □

4 Perfect Whole Edge Domination in Bipolar Fuzzy Graphs

Definition 4.1. A vertex v in a BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called a perfect bipolar fuzzy vertex if $\mu^P(v) = 1$ and $\mu^N(v) = -1$ (i.e.,) $\mu(v) = (1, -1)$ for all $v \in \mathcal{V}$.

Definition 4.2. An edge $e = v, w$ (simply vw) in a BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called a perfect bipolar fuzzy edge if $\rho^P(vw) = 1$ and $\rho^N(vw) = -1$ (i.e.,) $\rho(vw) = (1, -1)$ for all $vw \in E$

Proposition 4.1. *Every complete BFG is strong bipolar fuzzy graph.*

Proposition 4.2. *Every semi-complete bipolar fuzzy graph is semi-regular BFG.*

Definition 4.3. A vertex v in a whole edge domination BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called a perfect whole edge domination bipolar fuzzy vertex if $\mu^P(v) = 1$ and $\mu^N(v) = -1$ (i.e.,) $\mu(v) = (1, -1)$ for all $v \in V$.

Example 4.1. *Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a BFG, (Fig. 1:),*

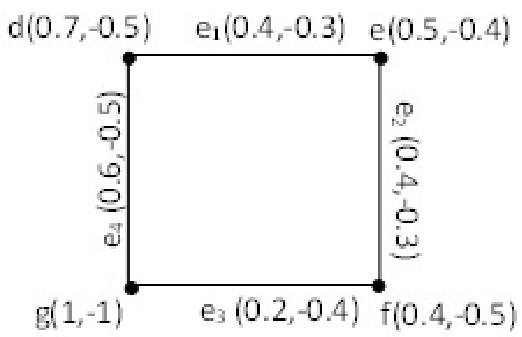

Fig. 7. Perfect whole edge domination bipolar fuzzy vertex in BFG

In this example, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a whole edge domination BFG, where $\mathcal{V} = \{d, e, f, g\}$ and $\mathcal{E}\{e_1, e_2, e_3, e_4\}$. Here $\mathcal{G}$ is the only perfect whole edge domination bipolar fuzzy vertex; here there is no perfect whole edge domination bipolar fuzzy vertex.

Definition 4.4. An edge $e = \{v, w\}$ (simply vw) in a whole edge domination BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called a perfect whole edge domination bipolar fuzzy edge if $\rho^P(vw) = 1$ and $\rho^N(vw) = -1$ (i.e.,) $\rho(vw) = (1, -1)$ for all $vw \in \mathcal{E}$

Example 4.2. *Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a BFG, (Fig. 2:),*

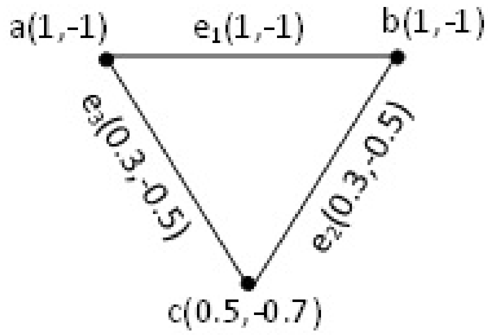

Fig. 8. Perfect whole edge domination bipolar fuzzy edge in BFG

In this example, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a whole edge domination BFG, where $\mathcal{V} = \{a, b, c\}$ and $\mathcal{E} = \{e_1, e_2, e_3\}$. Here e_1 is the only perfect whole edge domination bipolar fuzzy edge; but here there is no perfect whole edge domination bipolar fuzzy edge.

Definition 4.5. A whole edge domination BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called an μ-perfect whole edge domination BFG if all vertices in $\mathcal{G}$ are perfect whole edge domination bipolar fuzzy vertices.

Example 4.3. *Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a BFG, (Fig. 3:),*

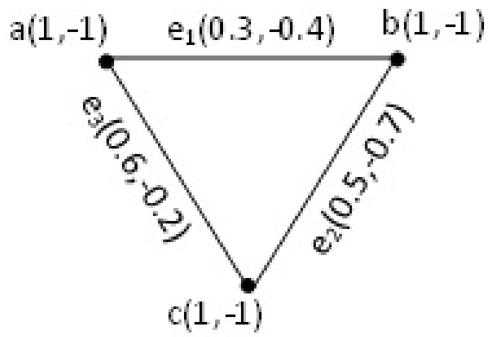

Fig. 9. μ-perfect whole edge domination BFG

Definition 4.6. A whole edge domination BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called an ρ-perfect whole edge domination BFG if all the edges in $\mathcal{G}$ are perfect whole edge domination bipolar fuzzy edges.

Example 4.4. *Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a BFG, (Fig. 4:),*

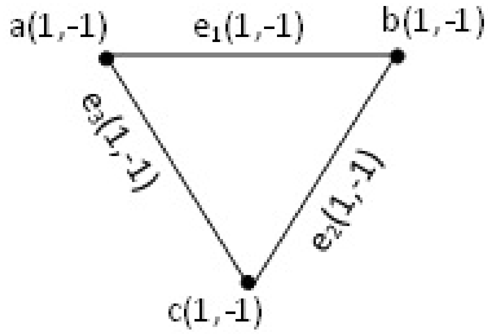

Fig. 10. μ-perfect and ρ-perfect whole edge domination BFG (1)

Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a whole edge domination BFG, (Fig. 5:),

Here all the vertices are perfect whole edge domination bipolar fuzzy vertices and all the edges are perfect whole edge domination bipolar fuzzy edges. Therefore, this is an μ-perfect and ρ-perfect whole edge domination BFG.

Proposition 4.3. *Every complete whole edge domination in bipolar fuzzy graph is strong whole edge domination BFG.*

Proposition 4.4. *Every ρ-perfect whole edge domination bipolar fuzzy graph is an μ-perfect whole edge domination BFG.*

Proof. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a ρ-perfect BFG. Then all edges in $\mathcal{G}$ have the bipolar fuzzy values (1,-1),
(i.e.,) $\rho(vw) = (1, -1)$ for all $vw \in \mathcal{E}$.

By the definition of BFG, we have $\rho(vw) = (\rho^P(vw), \rho^N(vw))$ where $\rho^P(vw) \leq min(\mu^P(v), \mu^P(w))$ and $\rho^N(vw) \leq min(\mu^N(v), \mu^N(w))$.

This implies that $1 \leq min(\mu^P(v), \mu^P(w))$ and $-1 \geq max(\mu^N(v), \mu^N(w))$.

Further, this implies that $\mu^P(v) = 1, \mu^P(w) = 1$, because greater than 1 is not possible; similarly $\mu^N(v) = -1, \mu^N(w) = -1$, because less than -1 is not possible.

Therefore $(\mu^P(v), \mu^N(v)) = (1, -1)$ and $(\mu^P(w), \mu^N(w)) = (1, -1)$, that is, $\mu(v) = (1, -1)$ and $\mu(w) = (1, -1)$.

In general, $\mu(v_i) = (1, -1)$ for all $v_i \in V$.

Hence, every ρ-perfect whole edge domination BFG is an μ-perfect whole edge domination BFG.
Note that the converse of this need not be true. □

Definition 4.7. A whole edge domination BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called a perfect bipolar fuzzy graph if it is an ρ-perfect whole edge domination BFG.

Definition 4.8. A whole edge domination BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called a complete perfect whole edge domination BFG if $\rho^P(vw) = 1$ and $\rho^N(vw) = -1$, (i.e.,) $\rho(vw) = (1, -1)$ for all $v, w \in \mathcal{V}$

Definition 4.9. A whole edge domination BFG $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is called semi-perfect whole edge domination BFG if all vertices in $\mathcal{G}$ are μ-perfect whole edge domination bipolar fuzzy vertices.

Theorem 4.5. *Every complete perfect whole edge domination BFG is a perfect whole edge domination BFG.*

Proof. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a complete perfect whole edge domination BFG. Since $\mathcal{G}$ is complete, all vertices are connected together. Since $\mathcal{G}$ is perfect, all edges are ρ-perfect. That is, $\rho(rw) = (1, -1)$, for all $vw \in \mathcal{E}$.

Therefore, obviously G is a perfect whole edge domination BFG.

Note that every perfect whole edge domination BFG is not necessarily a complete perfect whole edge domination BFG □

Theorem 4.6. *Every complete perfect whole edge domination bipolar fuzzy graph is a semi-perfect whole edge domination BFG.*

Proof. Since $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is complete perfect whole edge domination BFG, all edges are ρ-perfect and all vertices are joined by an edge. Clearly, $\mathcal{G}$ is μ-perfect. Therefore, $\mathcal{G}$ is a semi-perfect. Hence, every complete perfect whole edge domination bipolar fuzzy graph is a semi-perfect whole edge domination BFG. □

Proposition 4.5. *If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a perfect whole edge domination BFG, then (i) $d(v_i) = d_{\mathcal{E}}(vi) = d_N(vi)$, for all $v_i \in \mathcal{V}$; (ii) $\sum_{v_i} d(v_i) = \sum_{v_i} d_{\mathcal{E}}(vi) = \sum_{v_i} d_N(vi)$ for all $v_i \in \mathcal{V}$;*

Proof. Since $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is perfect whole edge domination BFG, it has μ-perfect and ρ-perfect, by definitions of degree, effective degree and neighbourhood degree, the results are immediate. □

Let us verify the Proposition by the following example.

Example 4.7. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a perfect whole edge domination BFG, where, $\mathcal{V} = \{v_1, v_2, v_3\}$ and $\mathcal{E} = \{e_1, e_2, e_3\}$ with $\mu(a) = \mu(b) = \mu(c) = (1, -1)$; $\rho(e_1) = \rho(e_2) = \rho(e_3) = (1, -1)$.*

This is an μ-perfect and ρ-perfect whole edge domination BFG.

By usual calculations, we get,
$d(v_1) = d_{\mathcal{E}}(v_1) = d_N(v_1) = (2, -2)$;
$d(v_2) = d_{\mathcal{E}}(v_2) = d_N(v_2) = (2, -2)$;
$d(v_3) = d_{\mathcal{E}}(v_3) = d_N(v_3) = (2, -2)$;
In general, $d(v_i) = d_{\mathcal{E}}(v_i) = d_N(v_i)$ *for all* $v_i \in \mathcal{V}$

Further, the sum of the degrees, sum of the effective degrees and sum of the neighbourhood degrees are same, that is,

$$\sum_{v_i} d(v_i) = \sum_{v_i} d_{\mathcal{E}}(v_i) = \sum_{v_i} d_N(v_i) = (6, -6) \text{ for all } v_i \in \mathcal{V}$$

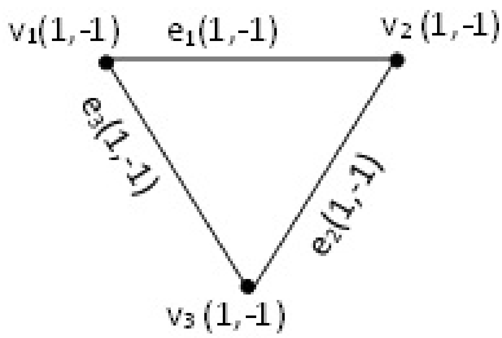

Fig. 11. **μ-perfect and ρ-perfect whole edge domination BFG(2)**

Theorem 4.8. *Every complete whole edge domination bipolar fuzzy graph is a totally regular whole edge domination BFG.*

Theorem 4.9. *Every complete perfect whole edge dominating bipolar fuzzy graph is regular whole edge dominating BFG.*

Proof. Since $\mathcal{G}$ is complete perfect whole edge dominating BFG with n vertices; the (open) neighbourhood degree of any vertex is $(n-1, -(n-1))$, i.e., $d_N(vi) = (n-1, -(n-1)) \times (1, -1)$, for all $v_i \in \mathcal{V}$. This is an $(n-1)$-regular BFG. Thus, every complete perfect whole edge dominating BFG is regular whole edge dominating BFG □

Proposition 4.6. *Every semi-perfect whole edge dominating bipolar fuzzy graph is not necessarily regular whole edge dominating BFG.*

Proposition 4.7. *In any complete perfect whole edge dominating BFG, $d(v_i) = d_{\mathcal{E}}(vi) = d_N(vi)$, for all $v_i \in \mathcal{V}$;*

5 Conclusion

In this work, we introduced whole edge domination in bipolar fuzzy graphs with some results. Some bounds and theorems were established as well. Further perfect whole edge domination in BFGs also introduced, related results and examples were also discussed.

Competing Interests

Authors have declared that no competing interests exist.

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