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Natural Vibrations of the Shallow Water Equation

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Abstract

The purpose of this paper is to construct the asymptotic for natural frequencies of the shallow water problem using the method of Wentzel-Kramers-Brillouin (WKB) and find the secular equation for the eigenvalues.

Keywords: Shallow water equation; Asymptotics; Eigenfunctions; Eigenvalues; Water waves; WKB method.

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1 Introduction

Many applications related to water waves involve shallow water equations. It includes dam break wave modeling, the breaking of waves on shallow beaches, tides in oceans, surges, flood waves in rivers and seiches in lakes. It can be found in [1], [2] and [3]. Therefore, this paper focuses on the WKB method for seiches. For the equation of the shallow water wave with a uniform small parameter, one can use the method (WKB) Wentzel - Kramers - Brillouin [4], also known in the literature as the approximation Liouville - Green. In [5], WKB method was used for finding asymptotic high frequency, this method is to obtain asymptotic series for solutions powers with a small parameter.

In the next section one takes the shallow water equation.

2 Mathematical Formulation

In this section we find the approximate solution as a linear combination of two linearly independent solutions. Substituting solution in boundary conditions, a homogeneous system of two equations is obtained. This system has non-trivial solutions when the determinant is zero.

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2.1 Shallow water equation

In [6], it is obtained the shallow water equation

$$\Phi_{tt} - g\nabla\big(d(x,z)\nabla\Phi\big) = 0, \tag{2.1}$$

where $\nabla = (\partial_x, \partial_z)$, g is the gravity acceleration, $d(x,z)$ is the water depth and Φ is the free surface elevation. The coordinates x and z are the horizontal coordinates. (See Figure 1 for the geometry of the problem)

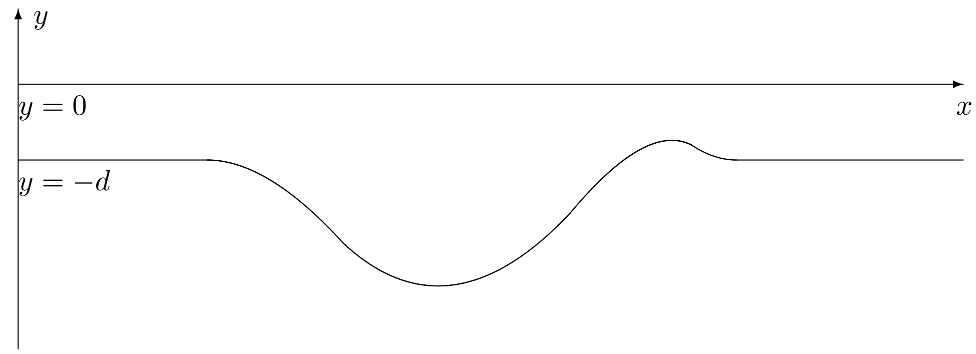

Figure 1: Geometry of the problem

Looking for the solution of shallow water equation in the form, $\Phi = \exp(i\omega t)\Psi(x,z)$, where ω is the frequency, for Ψ we obtain:

$$-\nabla(d\nabla\Psi) = \lambda\Psi, \qquad \lambda = \omega^2/g. \tag{2.2}$$

Assume that d describes the bottom that is parallel to the axis z: $d(x,z) = h_0 + V(x)$, and $V(x) \in C_0^\infty(\mathbb{R})$.

Looking for the solution of (2.2) in the form $\Psi = \exp(-ikz)\varphi(x)$, where k is the wavenumber. We obtain the equation

$$-h_0\varphi'' - (V\varphi')' + k^2(h_0 + V)\varphi = \frac{w^2}{g}\varphi, \tag{2.3}$$

where “prime” means derivative with respect to x.

The boundary value problem is

$$-h_0\varphi'' - (V\varphi')' + k^2(h_0 + V)\varphi = \frac{w^2}{g}\varphi. \tag{2.4}$$

$$\varphi'(0) = \varphi'(l) = 0. \tag{2.5}$$

The discrete spectrum of the problem (2.4)-(2.5) constitutes a sequence w_n of real numbers tending to infinity when $n \to \infty$. Hence, we can consider $\omega^2 = \frac{1}{\epsilon^2}$, where $\epsilon \to 0$ [5]. Therefore the problem (2.4)-(2.5) becomes

$$-h_0\varphi'' - (V\varphi')' + k^2(h_0+V)\varphi = \frac{1}{\varepsilon^2 g}\varphi. \tag{2.6}$$

$$\varphi'(0) = \varphi'(l) = 0. \tag{2.7}$$

In the next section we calculate the secular equation of the problem (2.6)–(2.7).

3 Main result

This section states and solves the problem of shallow water waves at high frequency, which is applied the WKB method.

The mathematical formulation of the problem is the searching for nontrivial solutions of the problem (2.6)–(2.7).

The main result is as follows

Theorem 3.1. *The eigenvalues of the problem* (2.6)*–(2.7) are given by* $w_n^2 = \frac{1}{\varepsilon_n^2}$*, with*

$$\omega_n = \left(\frac{n\pi}{\int_0^l a^{-\frac{1}{2}}(x)dx}\right)\left(1 + O\left(\frac{1}{n}\right)\right), \quad n = 0, 1, \ldots, \quad n \to \infty,$$

where $\varepsilon_n = \frac{L}{n\pi}$*,* $L = \int_0^l a^{-\frac{1}{2}}(t)dt$ *and function* $a^{-\frac{1}{2}}(x) = (g(h_0 + V(x)))^{-\frac{1}{2}}$*.*

Proof. Given that $\omega^2 = \frac{1}{\varepsilon^2}$ then equation (2.6) can be transformed into

$$\varphi = \varepsilon^2 g(-h_0\varphi'' - (V'\varphi' + V\varphi'') + k^2(h_0+V)\varphi). \tag{3.1}$$

Following the traditional WKB method, the analytical solution approximates equation (2.6) can be replaced by a power series given by the following

$$\varphi(x) = A(x,\varepsilon)e^{\frac{i\phi(x)}{\varepsilon}}, \quad \varepsilon \to 0, \tag{3.2}$$

where

$$A(x,\varepsilon) = A_0(x) + \varepsilon A_1(x) + \varepsilon^2 A_2(x) + \ldots, \quad \varepsilon \to 0, \tag{3.3}$$

with $\phi(x)$ and $A_j(x), j = 0, 1, 2, \ldots$ are smooth functions and unknown.

Replacing (3.2) and each of the derivatives of $v(x)$ in (3.1), we have the following expression

$$\begin{aligned} A(x,\varepsilon) =& g[(h_0+V(x))\phi_x^2(x)A(x,\varepsilon) - i\varepsilon(2(h_0+V(x))\phi_x(x)A_x(x,\varepsilon) \\ &+ (V_x(x)\phi_x(x) + (h_0+V(x))\phi_{xx}(x))A(x,\varepsilon))] + O(\varepsilon^2) \quad (3.4)\\ =& g[(h_0+V(x))\phi_x^2(x)A(x,\varepsilon) - i\varepsilon\big(2(h_0+V(x))\phi_x(x)A_x(x,\varepsilon) \\ &+ ((h_0+V(x))\phi_x(x))'A(x,\varepsilon))] + O(\varepsilon^2), \quad \varepsilon \to 0. \quad (3.5)\end{aligned}$$

Replacing (3.3) on both sides of (3.4) we obtain

$$\begin{aligned}
&A_0(x) + \varepsilon A_1(x) + \varepsilon^2 A_2(x) + \cdots = g\big[(h_0 + V(x))\phi_x^2(x)(A_0(x) + \varepsilon A_1(x) \\
&+\varepsilon^2 A_2(x) + \cdots) - i\varepsilon\big(2(h_0 + V(x))\phi_x(x)(A_{0x}(x) + \varepsilon A_{1x}(x) + \varepsilon^2 A_{2x}(x) + \cdots)\big) \\
&\quad + \big((h_0 + V(x))\phi_x(x)\big)'\big(A_0(x) + \varepsilon A_1(x) + \varepsilon^2 A_2(x) + \cdots\big)\big] + O(\varepsilon^2) \\
&= g(h_0 + V(x))\phi_x^2(x)A_0(x) + \varepsilon\Big[g(h_0 + V(x))\phi_x^2(x)A_1(x) \\
&\quad - 2ig(h_0 + V(x))\phi_x(x)A_{0x}(x) - ig\big((h_0 + V(x))\phi_x(x)\big)'A_0(x)\Big] \\
&\quad + O(\varepsilon^2), \quad \varepsilon \to 0. \qquad (3.6)
\end{aligned}$$

Equating the coefficients of the asymptotic series in ε and taking corresponding to ε^0 in (3.6) and using that $A_0 \neq 0$ as seen in the equation (3.10), it can be obtained

$$\varepsilon^0 : g(h_0 + V(x))\phi_x^2(x) = 1. \qquad (3.7)$$

From equation (3.7) and choosing the corresponding equality to ε^1 in (3.6), we obtain

$$\varepsilon^1 : 2(h_0 + V(x))\phi_x(x)A_{0x}(x) + ((h_0 + V(x))\phi_x(x))'A_0(x) = 0. \qquad (3.8)$$

By equating the asymptotic series, more equations are obtained. We consider only the first two ones, because other equations are of order $O(\varepsilon^2)$. Since equation (3.7) has two real roots with opposite signs, we obtain

$$\phi_k(x) = (-1)^k \int (g(h_0 + V(x)))^{-\frac{1}{2}}\, dx, \quad k = 1, 2. \qquad (3.9)$$

From the equation (3.8) and separating the functions $A_0(x)$, $(h_0 + V(x))\phi_x(x)$ and integrating on both sides, it follows that

$$A_0(x) = C((h_0 + V(x))\phi_x(x))^{-\frac{1}{2}}, \qquad (3.10)$$

where C is a non-zero arbitrary constant. Therefore, differentiating with respect to x in the equation (3.9) and substituting (3.10), function $A_0(x)$ can be expressed as follows

$$A_0(x) = C(h_0 + V(x))^{-\frac{1}{4}}. \qquad (3.11)$$

Therefore, replacing (3.9) and (3.11) in (3.2) which is the solution $v(x)$ of (2.6), we have

$$\varphi_1(x) = d_1(h_0 + V(x))^{-1/4} \sin\left(\frac{\psi_1(x)}{\varepsilon}\right) + O(\varepsilon), \quad \varepsilon \to 0, \qquad (3.12)$$

$$\varphi_2(x) = d_2(h_0 + V(x))^{-1/4} \cos\left(\frac{\psi_1(x)}{\varepsilon}\right) + O(\varepsilon), \quad \varepsilon \to 0, \qquad (3.13)$$

and writing the solution (2.6) as the first term of the linear combination of $v_1(x), v_2(x)$

$$\varphi(x) = (h_0 + V(x))^{-1/4}\left(c_1 \cos\left(\frac{\psi_1(x)}{\varepsilon}\right) + c_2 \sin\left(\frac{\psi_1(x)}{\varepsilon}\right)\right), \qquad (3.14)$$

$\varepsilon \to 0$, where

$$\psi_1(x) = \int_0^x a^{-\frac{1}{2}}(t)dt, \quad a^{-\frac{1}{2}}(x) = (g(h_0 + V(x)))^{-\frac{1}{2}}, \qquad (3.15)$$

and c_1, c_2 are constants. It is noted that $\sin\left(\frac{\psi_1}{\varepsilon}\right), \cos\left(\frac{\psi_1}{\varepsilon}\right)$, are linearly independent.

Solution (3.14) and boundary conditions (2.5) yield a homogeneous system of two equations for two constants $c_i, i = 1, 2$. This system has nontrivial solutions when

$$\begin{vmatrix} 0 & 1 \\ -\sin\left(\frac{L}{\varepsilon}\right) & \cos\left(\frac{L}{\varepsilon}\right) \end{vmatrix} = 0, \tag{3.16}$$

where

$$L = \int_0^l a^{-\frac{1}{2}}(t)dt. \tag{3.17}$$

The equation (3.16) is the secular equation for natural frequency $\omega_n = \varepsilon_n^{-2}$.
Therefore

$$\sin\left(\frac{L}{\varepsilon}\right) = 0, \quad \varepsilon \to 0, \quad \varepsilon = \varepsilon_n = \frac{L}{n\pi} \tag{3.18}$$

then

$$\omega_n = \left(\frac{n\pi}{\int_0^l a^{-\frac{1}{2}}(t)dt}\right)\left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty, n = 0, 1, \ldots.$$

From (3.18) and initial conditions, it follows the eigenfunction is

$$\varphi_n(x) = c_3(h_0 + V(x))^{-1/4}\cos\left(\frac{\psi_1(x)}{\varepsilon_n}\right),$$

where c_3 is an arbitrary constant. □

4 CONCLUSIONS

This paper contains three sections providing several new ideas in the theory of shallow water waves.

a In the section (1), we have introduced seiches and WKB method.

b In the section (2), we have studied the shallow water equation for seiches.

c In the section (3), we have obtained the eigenvalues and eigenfunctions that appear in seiches.

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Competing interests

The authors declare that no competing interests exist.

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