Existence result for a singular semipositone dynamic system on time scales

Arzu Denk Oguz[1] and Fatma Serap Topal[2,*]

[1]   Department of Mathematics, Atilim University, Incek, Ankara-Turkey.; arzu.denk@hotmail.com
[2]   Department of Mathematics, Ege University, Bornova, Izmir-Turkey.
*   Correspondence: f.serap.topal@ege.edu.tr

Abstract: We concentrate on investigating the existence of positive solutions for the system of second order singular semipositone m-point boundary value problems in this article. We emphasize that the nonlinear term may take a negative value and be singular. By the properties of Green's function and applying fixed point theorem in cones, existence results for positive solutions are obtained. Also, we provide an example to make our results clear and easy for readers to understand the existence result.

1. Introduction

Multi-point boundary value problems for second order and higher order ordinary differential equations and systems arise from many fields in physics, biology and chemistry. These problems play very important role in both theory and applications [1–5].

Problems where the nonlinear terms have some singularities are referred to as singular problems in the literature and this type of differential systems appear in the study of gas dynamics, fluid mechanics, in the theory of boundary layer and so on. Because of its applications in physics, singular problems have extensively study in recent years, for example see [6–9].

For example, Asif and Khan [6] studied the existence of positive solution to a nonlinear singular system with four-point boundary conditions of the type

$$
\begin{aligned}
-x(t)'' &= f(t, x(t), y(t)), \quad t \in (0,1), \\
-y(t)'' &= g(t, x(t), y(t)), \quad t \in (0,1), \\
x(0) &= 0, \ x(1) = \alpha y(\xi), \ y(0) = 0, \ y(1) = \beta x(\eta).
\end{aligned}
$$

In [7], Liu and Yan considered the following singular boundary value problem of Sturm Liouville differential system:

$$
\begin{aligned}
(p(t)x(t)')' + \lambda f(t, x(t), y(t)) &= 0, \quad t \in (0,1), \\
(p(t)y(t)')' + \lambda g(t, x(t), y(t)) &= 0, \quad t \in (0,1), \\
\alpha x(0) - \beta p(0)x'(0) &= \gamma x(1) - \delta p(1)x'(1) = 0, \\
\alpha y(0) - \beta p(0)y'(0) &= \gamma y(1) - \delta p(1)y'(1) = 0.
\end{aligned}
$$

Although much interest has been observed in investigating the existence of positive solutions of dynamic equations on measure chains [9–13], very few research articles has been seen on the existence of positive solutions of dynamic systems on measure chains [14,15].

In [16], Prasad, Rao and Bharathi interested in the existence of positive solutions to the system of dynamic equations:

$$
(-1)^n u^{\triangle^{2n}}(t) + \lambda p(t) f(v(\sigma(t))) = 0, \quad t \in [a, b],
$$

$$(-1)^n v^{\triangle^{2n}}(t) + \mu q(t) f(u(\sigma(t))) = 0, \quad t \in [a,b],$$
$$\alpha_{i+1} u^{\triangle^{2i}}(a) - \beta_{i+1} u^{\triangle^{2i+1}}(a) = 0, \ \gamma_{i+1} u^{\triangle^{2i}}(\sigma(b)) + \delta_{i+1} u^{\triangle^{2i+1}}(\sigma(b)) = 0,$$
$$\alpha_{i+1} v^{\triangle^{2i}}(a) - \beta_{i+1} v^{\triangle^{2i+1}}(a) = 0, \ \gamma_{i+1} v^{\triangle^{2i}}(\sigma(b)) + \delta_{i+1} v^{\triangle^{2i+1}}(\sigma(b)) = 0.$$

Problems of this type where the nonlinear term may change sign are referred to as semipositone problems in the literature. Semipositone differential systems appear in the study of chemical reactors [17].

The above works motivates us to consider the nonlinear singular semipositone system of *m*-point boundary value problem (SSS) in this paper.

$$\begin{cases} -[p(t)u_i^{\triangle}(t)]^{\nabla} + q(t)u_i(t) = f_i(t, u_1(t), u_2(t)) + h_i(t), & t \in (a,b), \quad i = 1,2, \\ \alpha u_i(a) - \beta u_i^{[\triangle]}(a) = \sum_{k=1}^{m-2} \alpha_k u_i(\xi_k), \quad \gamma u_i(b) + \delta u_i^{[\triangle]}(b) = \sum_{k=1}^{m-2} \beta_k u_i(\xi_k), \quad i = 1,2, \end{cases} \tag{1}$$

where $\alpha, \beta, \gamma, \delta, \xi_k, \alpha_k, \beta_k$ (for $k \in \{1, 2, ..., m-2\}$) are complex constants such that $|\alpha| + |\beta| \neq 0$, $|\gamma| + |\delta| \neq 0$ and $\xi_k \in \mathrm{T} \backslash \{a, b\}$, $q : \mathrm{T} \to \mathcal{C}$ is a continuous function, $p : \mathrm{T} \to \mathcal{C}$ is $\nabla-$ differentiable on T_k, $p(t) \neq 0$ for all $t \in \mathrm{T}$, $p^{\nabla} : \mathrm{T}_k \to \mathcal{C}$ is continuous, f_1 and $f_2 : (a,b) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and may be singular at $t = a, b$ and h_1 and $h_2 : (a,b) \to (-\infty, \infty)$ are Lebesgue integrable and may have finitely many singularities in $[a, b]$.

By an interval (a, b), we mean the intersection of the real interval (a, b) with the given time scale T. Some preliminary definitions and theorems on time scales can be found in the books [18,19].

Compared to previous work in this field, this study presented three new features. Firstly, the nonlinear term is allowed to change sign and tends to negative infinity. Secondly, is allowed to have finitely many singularities in $[a, b]$. Lastly, the boundary condition taken up generalizes the conditions of many problems in the literature. By using the cone theory technique, we establish some sufficient conditions for the existence of multiple positive solutions to the SSS (1). The rest of the paper is organized as follows: Section 2 gives some inequalities for Green's function and some results which are needed later. Criteria for the existence of positive solutions of the SSS (1) is established in Section 3 and uses fixed point index theorem. In addition, an example is given to illustrate the applications of main result.

2. The preliminary Lemmas

We shall work in the space $E = C([a,b]; R) \times C([a,b]; R)$. The space E is a *Banach space* if it is endowed with the norm as follows:

$$\|(u_1, u_2)\| = \|u_1\| + \|u_2\|, \quad \|u_i\| = \max_{t \in [a,b]} |u_i(t)|, \quad i = 1,2,$$

for any $(u_1, u_2) \in E$. For any $u = (u_1, u_2), v = (v_1, v_2) \in E$, we denote

$$u \leq v \Leftrightarrow u_i(t) \leq v_i(t), \quad t \in [a,b], \quad i = 1,2.$$

In the following, let us define a cone P of E by

$$P = \{(u_1, u_2) \in E : u_i(t) \geq g(t)\|u_i\|, \quad t \in [a,b], \quad i = 1,2\},$$

where g is defined by

$$g(t) := \min_{t \in [a,b]} \left\{ \frac{\phi_1(t)}{\phi_1(b)}, \frac{\phi_2(t)}{\phi_2(a)} \right\}, \tag{2}$$

and ϕ_1, ϕ_2 are the solutions of the linear problems

$$[p(t)\phi_1^{\triangle}(t)]^{\nabla} - q(t)\phi_1(t) = 0, \quad t \in (a,b),$$
$$\phi_1(a) = \beta, \quad \phi_1^{[\triangle]}(a) = \alpha,$$

and

$$[p(t)\phi_2^\triangle(t)]^\nabla - q(t)\phi_2(t) = 0, \quad t \in (a,b),$$
$$\phi_2(b) = \delta, \quad \phi_2^{[\triangle]}(b) = -\gamma,$$

respectively. Let $G(t,s)$ be the Green's function for the boundary value problem

$$-[p(t)u^\triangle(t)]^\nabla + q(t)u(t) = 0, \quad t \in (a,b),$$
$$\alpha u(a) - \beta u^{[\triangle]}(a) = 0, \quad \gamma u(b) + \delta u^{[\triangle]}(b) = 0,$$

is given by

$$G(t,s) := \frac{1}{d} \begin{cases} \phi_1(s)\phi_2(t), & a \le s \le t \le b, \\ \phi_1(t)\phi_2(s), & a \le t \le s \le b, \end{cases} \tag{3}$$

where $d = -W_t(\phi_1,\phi_2) = p(t)[\phi_1^\triangle(t)\phi_2(t) - \phi_1(t)\phi_2^\triangle(t)]$.

Let us define

$$\Omega := \begin{vmatrix} -\sum_{k=1}^{m-2} \alpha_k \phi_1(\xi_k) & d - \sum_{k=1}^{m-2} \alpha_k \phi_2(\xi_k) \\ d - \sum_{k=1}^{m-2} \beta_k \phi_1(\xi_k) & -\sum_{k=1}^{m-2} \beta_k \phi_2(\xi_k) \end{vmatrix}$$

and assume that the following conditions are satisfied:

(H_1) $p(t) > 0, q(t) \ge 0,$
(H_2) $\alpha, \gamma \ge 0, \beta, \delta > 0, \alpha_k, \beta_k \ge 0$ for $k \in \{1, 2, ..., m-2\}$,
(H_3) If $q(t) \equiv 0$, then $\alpha + \gamma > 0$,
(H_4) $\Omega < 0, d - \sum_{k=1}^{m-2} \alpha_k \phi_2(\xi_k) > 0, d - \sum_{k=1}^{m-2} \beta_k \phi_1(\xi_k) > 0.$

To prove the main results, we will employ following lemmas.

Lemma 1. *[18] Under the conditions (H_1) and (H_2), the solutions $\phi_1(t)$ and $\phi_2(t)$ posses the following properties:*

$$\phi_1(t), \phi_2(t) \ge 0, \quad \phi_1^{[\triangle]}(t) \ge 0, \quad \phi_2^{[\triangle]}(t) \le 0, \quad t \in [a,b].$$

Lemma 2. *[18] If the conditions $(H_1) - (H_3)$ are hold, then $G(t,s) > 0$ for $t, s \in [a,b]$.*

Lemma 3. *[20] Assume that $(H_1) - (H_3)$ hold. Then*

$$g(t)G(s,s) \le G(t,s) \le G(s,s), \quad t, s \in [a,b],$$

where g is given in equation (2).

We consider the following boundary value problem

$$-[p(t)u^\triangle(t)]^\nabla + q(t)u(t) = y(t), \quad t \in (a,b), \tag{4}$$

$$\alpha u(a) - \beta u^{[\triangle]}(a) = \sum_{k=1}^{m-2} \alpha_k u(\xi_k), \quad \gamma u(b) + \delta u^{[\triangle]}(b) = \sum_{k=1}^{m-2} \beta_k u(\xi_k). \tag{5}$$

Lemma 4. *[12] Let the conditions $(H_1) - (H_3)$ be hold. Assume that $\Omega :\neq 0$. Then for $y \in C([a,b])$, the boundary value problem given in equations (4)-(5) has a unique solution*

$$u(t) = \int_a^b G(t,s)y(s)\nabla s + A(y)\phi_1(t) + B(y)\phi_2(t),$$

where $G(t,s)$ is given in equation (3),

$$A(y) := \frac{1}{\Omega} \begin{vmatrix} \sum_{k=1}^{m-2} \alpha_k \int_a^b G(\xi_k,s)y(s)\nabla s & d - \sum_{k=1}^{m-2} \alpha_k \phi_2(\xi_k) \\ \sum_{k=1}^{m-2} \beta_k \int_a^b G(\xi_k,s)y(s)\nabla s & -\sum_{k=1}^{m-2} \beta_k \phi_2(\xi_k) \end{vmatrix},$$

$$B(y) := \frac{1}{\Omega} \left| \begin{array}{cc} -\sum_{k=1}^{m-2} \alpha_k \phi_1(\xi_k) & \sum_{k=1}^{m-2} \alpha_k \int_a^b G(\xi_k,s)y(s)\nabla s \\ d - \sum_{k=1}^{m-2} \beta_k \phi_1(\xi_k) & \sum_{k=1}^{m-2} \beta_k \int_a^b G(\xi_k,s)y(s)\nabla s \end{array} \right|.$$

Lemma 5. *[5] Let $(H_1) - (H_4)$ hold. If $y \in C([a,b],[0,\infty))$, then the solution u of the boundary value problem (4)-(5) satisfies $u(t) \geq 0$, for $t \in [a,b]$.*

Lemma 6. *If $\int_a^b G(s,s)y(s)\nabla s < \infty$, then the following inequalities are satisfied:*

$$A(y) \leq A \int_a^b G(s,s)y(s)\nabla s, \quad B(y) \leq B \int_a^b G(s,s)y(s)\nabla s,$$

where

$$A = \frac{1}{\Omega} \left| \begin{array}{cc} \sum_{k=1}^{m-2} \alpha_k & d - \sum_{k=1}^{m-2} \alpha_k \phi_2(\xi_k) \\ \sum_{k=1}^{m-2} \beta_k & -\sum_{k=1}^{m-2} \beta_k \phi_2(\xi_k) \end{array} \right|,$$

$$B = \frac{1}{\Omega} \left| \begin{array}{cc} -\sum_{k=1}^{m-2} \alpha_k \phi_1(\xi_k) & \sum_{k=1}^{m-2} \alpha_k \\ d - \sum_{k=1}^{m-2} \beta_k \phi_1(\xi_k) & \sum_{k=1}^{m-2} \beta_k \end{array} \right|.$$

3. Main result

In this section, we apply the following fixed point index theorem to prove the existence of at least one positive solution for the SSS (1).

Theorem 1. *Let $E = (E, \|.\|)$ be a Banach space, Ω be a bounded open subset of E with $0 \in \Omega$, $P \subset E$ be a cone in E and $F : P \cap \overline{\Omega} \to P$ be a completely continuous operator.*

(i) *Suppose that $Fu \neq \lambda u, \quad \forall u \in \partial\Omega \cap P, \lambda \geq 1$. Then $i(F, \Omega \cap P, P) = 1$.*
(ii) *Suppose that $Fu \nleq u, \quad \forall u \in \partial\Omega \cap P$. Then $i(F, \Omega \cap P, P) = 0$.*

In the remaining part of the paper, we assume that the following conditions are satisfied:

(H_5) $f_1, f_2 : (a,b) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous,
(H_6) $h_1, h_2 : (a,b) \to (-\infty,\infty)$ are Lebesgue integrable such that $0 < \int_a^b G(s,s)[f_i(s,1,1) + h_{i_+}(s)]\nabla s < \infty$
    and $\int_a^b h_{i_-}(s)\nabla s > 0 \quad i = 1,2$,
    where $h_{i_+}(t) = \max\{h_i(t),0\}$ and $h_{i_-}(t) = \max\{-h_i(t),0\}$.

Remark 1. By the assumption (H_6), we have $\int_a^b G(t,s)h_{i_-}(s)\nabla s < \infty, \quad i = 1,2$.

In fact, from the properties of ϕ_1, ϕ_2 and Green function, we get

$$\int_a^b G(t,s)h_{i_-}(s)\nabla s \leq \int_a^b G(s,s)h_{i_-}(s)\nabla s \leq \phi_1(b)\phi_2(a) \int_a^b h_{i_-}(s)\nabla s < \infty, \quad i = 1,2.$$

Let $w_i(t) = \int_a^b G(t,s)h_{i_-}(s)\nabla s + A(h_{i_-})\phi_1(t) + B(h_{i_-})\phi_2(t)$, $t \in [a,b]$, $i = 1,2$. Using the expression for Green's function, the definition of the function g, the properties of ϕ_1 and ϕ_2, the assumption (H_6) and Lemma 6, we obtain

$$w_i(t) = \int_a^b G(t,s)h_{i_-}(s)\nabla s + A(h_{i_-})\phi_1(t) + B(h_{i_-})\phi_2(t)$$

$$= \frac{1}{d} \int_a^t \phi_1(s)\phi_2(t)h_{i_-}(s)\nabla s + \frac{1}{d} \int_t^b \phi_1(t)\phi_2(s)h_{i_-}(s)\nabla s + A(h_{i_-})\phi_1(t) + B(h_{i_-})\phi_2(t)$$

$$\leq \frac{1}{d} \int_a^t \phi_1(t)\phi_2(t)h_{i_-}(s)\nabla s + \frac{1}{d} \int_t^b \phi_1(t)\phi_2(t)h_{i_-}(s)\nabla s + A\phi_1(t) \int_a^b G(s,s)h_{i_-}(s)\nabla s$$

$$\quad + B\phi_2(t) \int_a^b G(s,s)h_{i_-}(s)\nabla s$$

$$= \frac{1}{d} \int_a^b \phi_1(t)\phi_2(t)h_{i_-}(s)\nabla s + A\phi_1(t) \int_a^b G(s,s)h_{i_-}(s)\nabla s + B\phi_2(t) \int_a^b G(s,s)h_{i_-}(s)\nabla s$$

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$$\leq \frac{1}{d}\phi_1(b)\phi_2(a)g(t)\int_a^b h_{i_-}(s)\nabla s + \frac{1}{\phi_2(t)}A\phi_1(b)\phi_2(a)g(t)\int_a^b G(s,s)h_{i_-}(s)\nabla s$$

$$+ \frac{1}{\phi_1(t)}B\phi_1(b)\phi_2(a)g(t)\int_a^b G(s,s)h_{i_-}(s)\nabla s$$

$$\leq \frac{1}{d}\phi_1(b)\phi_2(a)g(t)\int_a^b h_{i_-}(s)\nabla s + \frac{1}{\phi_2(b)}A\phi_1(b)\phi_2(a)g(t)\int_a^b G(s,s)h_{i_-}(s)\nabla s$$

$$+ \frac{1}{\phi_1(a)}B\phi_1(b)\phi_2(a)g(t)\int_a^b G(s,s)h_{i_-}(s)\nabla s$$

$$= \Big[\frac{1}{d}\int_a^b h_{i_-}(s)\nabla s + \Big(\frac{A}{\phi_2(b)} + \frac{B}{\phi_1(a)}\Big)\int_a^b G(s,s)h_{i_-}(s)\nabla s\Big]\phi_1(b)\phi_2(a)g(t) < +\infty, \quad i=1,2.$$

Therefore, we can write

$$w_i(t) \leq C_i g(t), \quad t \in [a,b], \quad i=1,2, \tag{6}$$

where

$$C_i = \Big[\frac{1}{d}\int_a^b h_{i_-}(s)\nabla s + \Big(\frac{A}{\phi_2(b)} + \frac{B}{\phi_1(a)}\Big)\int_a^b G(s,s)h_{i_-}(s)\nabla s\Big]\phi_1(b)\phi_2(a)$$

and g is given in equation (2). Therefore, $w_i(t), i=1,2$ are well defined in E. By direct computation, we have

$$\alpha w_i(a) - \beta w_i^{[\triangle]}(a) = \sum_{k=1}^{m-2}\alpha_k w_i(\xi_k), \quad \gamma w_i(b) + \delta w_i^{[\triangle]}(b) = \sum_{k=1}^{m-2}\beta_k w_i(\xi_k), \quad i=1,2,$$

which implies that $w_i(t), i=1,2$ are positive solutions of the following boundary value problems:

$$- [p(t)u_1^{\triangle}(t)]^\nabla + q(t)u_1(t) = h_{1_-}(t), \quad t \in (a,b),$$

$$\alpha u_1(a) - \beta u_1^{[\triangle]}(a) = \sum_{k=1}^{m-2}\alpha_k u_1(\xi_k), \quad \gamma u_1(b) + \delta u_1^{[\triangle]}(b) = \sum_{k=1}^{m-2}\beta_k u_1(\xi_k),$$

and

$$- [p(t)u_2^{\triangle}(t)]^\nabla + q(t)u_2(t) = h_{2_-}(t), \quad t \in (a,b),$$

$$\alpha u_2(a) - \beta u_2^{[\triangle]}(a) = \sum_{k=1}^{m-2}\alpha_k u_2(\xi_k), \quad \gamma u_2(b) + \delta u_2^{[\triangle]}(b) = \sum_{k=1}^{m-2}\beta_k u_2(\xi_k),$$

respectively.

For any $u(t) \in C([a,b])$, let us define a function $[.]^*$ by

$$[u(t)]^* = \begin{cases} u(t), & u(t) \geq 0, \\ 0 & u(t) < 0. \end{cases}$$

Now, we consider the following dynamic system

$$\begin{cases} -[p(t)u_i^{\triangle}(t)]^\nabla + q(t)u_i(t) = f_i(t, [(u_1-w_1)(t)]^*, [(u_2-w_2)(t)]^*) + h_{i_+}(t), & t \in (a,b), \quad i=1,2, \\ \alpha u_i(a) - \beta u_i^{[\triangle]}(a) = \sum_{k=1}^{m-2}\alpha_k u_i(\xi_k), \quad \gamma u_i(b) + \delta u_i^{[\triangle]}(b) = \sum_{k=1}^{m-2}\beta_k u_i(\xi_k), & i=1,2, \end{cases} \tag{7}$$

and we define the operator $F : E \to E$ by

$$F(u_1,u_2) = (F_1(u_1,u_2), F_2(u_1,u_2))$$

where

$$F_i(u_1,u_2)(t) = \int_a^b G(t,s)[f_i(s, [u_1(s)-w_1(s)]^*, [u_2(s)-w_2(s)]^*) + h_{i_+}(s)]\nabla s + A(f_i + h_{i_+})\phi_1(t)$$

$$+ B(f_i + h_{i_+})\phi_2(t), \quad i=1,2.$$

It is well known that the existence of the solution to the system (7) is equivalent to the existence of fixed point of the operator F. Therefore, we shall seek a fixed point of F in our cone P.

Lemma 7. *If (v_1, v_2) with $(w_1, w_2) \leq (v_1, v_2)$ is a positive solution of the system (7), then $(v_1 - w_1, v_2 - w_2)$ is a positive solution of the SSS (1).*

Proof. Suppose that (v_1, v_2) with $(w_1, w_2) \leq (v_1, v_2)$ is a positive solution of system (7), then from (7) and the definition of $[.]^*$, we have

$$\begin{cases} -[p(t)v_i^{\triangle}(t)]^{\nabla} + q(t)v_i(t) = f_i(t, v_1(t) - w_1(t), v_2(t) - w_2(t)) + h_{i_+}(t), & t \in (a,b), \quad i = 1,2, \\ \alpha v_i(a) - \beta v_i^{[\triangle]}(a) = \sum_{k=1}^{m-2} \alpha_k v_i(\xi_k), \quad \gamma v_i(b) + \delta v_i^{[\triangle]}(b) = \sum_{k=1}^{m-2} \beta_k v_i(\xi_k), \quad i = 1,2. \end{cases} \tag{8}$$

Let $u_i(t) = v_i(t) - w_i(t), i = 1, 2$, then $v_i(t) = u_i(t) + w_i(t)$, $v_i^{\triangle}(t) = (u_i + w_i)^{\triangle}(t) = u_i^{\triangle}(t) + w_i^{\triangle}(t)$ and

$$[p(t)v_i^{\triangle}(t)]^{\nabla} = [p(t)(u_i^{\triangle}(t) + w_i^{\triangle}(t))]^{\nabla} = [p(t)u_i^{\triangle}(t)]^{\nabla} + [p(t)w_i^{\triangle}(t)]^{\nabla},$$

thus (8) becomes

$$\begin{cases} -[p(t)u_i^{\triangle}(t)]^{\nabla} + q(t)u_i(t) = f_i(t, u_1(t), u_2(t)) + h_{i_+}(t) - h_{i_-}(t), & t \in (a,b), \quad i = 1,2, \\ \alpha u_i(a) - \beta u_i^{[\triangle]}(a) = \sum_{k=1}^{m-2} \alpha_k u_i(\xi_k), \quad \gamma u_i(b) + \delta u_i^{[\triangle]}(b) = \sum_{k=1}^{m-2} \beta_k u_i(\xi_k), \quad i = 1,2, \end{cases} \tag{9}$$

Notice that $h_i(t) = h_{i_+}(t) - h_{i_-}(t), i = 1, 2$ and (9). We know that $(u_1, u_2) = (v_1 - w_1, v_2 - w_2)$ is a positive solution of the SSS (1). This completes the proof. $\square$

Now, we want to give the main result of this paper. To prove the main theorem, we need the following assumptions for the functions f_i, i= 1, 2.

(H_7) For $t \in (a,b)$, $f_i(t, 1, 1) \neq 0$ $(i = 1, 2)$, there exists constants $\lambda_1 \geq \lambda_2 > 1$ such that, for $t \in (a,b)$, $u_1, u_2 \in [0, \infty)$,

$$(c_1 c_2)^{\lambda_1} f_i(t, u_1, u_2) \leq f_i(t, c_1 u_1, c_2 u_2) \leq (c_1 c_2)^{\lambda_2} f_i(t, u_1, u_2), \quad \forall c_1, c_2 \in [0, 1], \quad i = 1, 2.$$

Remark 2. For $c_1, c_2 \geq 1$, $(t, u_1, u_2) \in (a,b) \times [0, \infty) \times [0, \infty)$, we have

$$(c_1 c_2)^{\lambda_2} f_i(t, u_1, u_2) \leq f_i(t, c_1 u_1, c_2 u_2) \leq (c_1 c_2)^{\lambda_1} f_i(t, u_1, u_2), \quad i = 1, 2.$$

In fact, from the assumption (H_7), for $c_1, c_2 \geq 1$, $(t, u_1, u_2) \in (a,b) \times [0, \infty) \times [0, \infty)$, we get

$$f_i(t, u_1, u_2) = f_i(t, \frac{1}{c_1} c_1 u_1, \frac{1}{c_2} c_2 u_2) \leq (\frac{1}{c_1 c_2})^{\lambda_2} f_i(t, c_1 u_1, c_2 u_2),$$

This implies

$$(c_1 c_2)^{\lambda_2} f_i(t, u_1, u_2) \leq f_i(t, c_1 u_1, c_2 u_2), \quad i = 1, 2.$$

At the same time, we have

$$f_i(t, c_1 u_1, c_2 u_2) \leq (c_1 c_2)^{\lambda_1} f_i(t, u_1, u_2), \quad i = 1, 2.$$

Therefore, when $c_1, c_2 \geq 1$, we have

$$(c_1 c_2)^{\lambda_2} f_i(t, u_1, u_2) \leq f_i(t, c_1 u_1, c_2 u_2) \leq (c_1 c_2)^{\lambda_1} f_i(t, u_1, u_2), \quad i = 1, 2.$$

Lemma 8. *If $f_i(t, u_1, u_2)(i = 1, 2)$ satisfies (H_7), then for $(t, u_1, u_2) \in (a,b) \times [0, \infty) \times [0, \infty)$, $f_i(t, u_1, u_2)$ is increasing on u_1, u_2 and for $[t_1, t_2] \subset (a,b)$,*

$$\lim_{u_1, u_2 \to +\infty} \min_{t \in [t_1, t_2]} \frac{f_i(t, u_1, u_2)}{|u_1| + |u_2|} = +\infty, \quad i = 1, 2.$$

Proof. Let $t \in (a, b), u_1, v_1, u_2 \in [0, \infty)$ such that $u_1 \leq v_1$. We will show that $f_i(t, u_1, u_2) \leq f_i(t, v_1, u_2)(i = 1, 2)$. Clearly, if $v_1 = 0$, then $f_i(t, u_1, u_2) \leq f_i(t, v_1, u_2)$. If $v_1 \neq 0$, let $a_1 = u_1/v_1$, then $0 \leq a_1 \leq 1$. Now, using the assumption (H_7), we obtain

$$f_i(t, u_1, u_2) = f_i(t, a_1 v_1, u_2) \leq a_1^{\lambda_2} f_i(t, v_1, u_2) \leq f_i(t, v_1, u_2), \quad i = 1, 2.$$

Thus, we get that $f_i(t, u_1, u_2)$ is increasing on u_1. Similarly, we can prove that $f_i(t, u_1, u_2)$ is increasing on u_2. On the other hand, choose $u_1, u_2 > 1$. Considering the Remark 2, we get

$$f_i(t, u_1, u_2) \geq (u_1 u_2)^{\lambda_2} f_i(t, 1, 1), \quad i = 1, 2,$$

and thus, for $[t_1, t_2] \subset (a, b), \forall t \in [t_1, t_2]$, we have

$$\min_{t \in [t_1, t_2]} \frac{f_i(t, u_1, u_2)}{|u_1| + |u_2|} \geq \min_{t \in [t_1, t_2]} \frac{(u_1 u_2)^{\lambda_2}}{|u_1| + |u_2|} f_i(t, 1, 1) > 0,$$

Therefore, we obtain

$$\lim_{u_1, u_2 \to +\infty} \min_{t \in [t_1, t_2]} \frac{f_i(t, u_1, u_2)}{|u_1| + |u_2|} = +\infty, \quad i = 1, 2.$$

□

Lemma 9. *Assume that $(H_1) - (H_7)$ hold. Then $F : P \to P$ is a completely continuous operator.*

Proof. First, we shall show that the operator $F : P \to P$ is well defined. Therefore, for any fixed $(u_1, u_2) \in P$, choose $0 < d_1, d_2 < 1$ such that $d_1 \|u_1\| < 1$ and $d_2 \|u_2\| < 1$. Then for $t \in [a, b]$, we get

$$d_i[u_i(t) - w_i(t)]^* \leq d_i u_i(t) \leq d_i \|u_i\| < 1, \quad i = 1, 2.$$

Thus, using Remark 2 and Lemma 8, we get

$$f_i(t, [u_1(t) - w_1(t)]^*, [u_2(t) - w_2(t)]^*) \leq (\frac{1}{d_1 d_2})^{\lambda_1} f_i(t, d_1 \|u_1\|, d_2 \|u_2\|)$$
$$\leq (d_1 d_2)^{\lambda_2 - \lambda_1} \|u_1\|^{\lambda_2} \|u_2\|^{\lambda_2} f_i(t, 1, 1), \quad i = 1, 2,$$

from which the assumption (H_6), the properties of ϕ_1, ϕ_2 and Lemma 6, for any $t \in [a, b]$ gives us:

$$\begin{aligned}
F_i(u_1, u_2)(t) &= \int_a^b G(t, s)[f_i(s, [u_1(s) - w_1(s)]^*, [u_2(s) - w_2(s)]^*) + h_{i_+}(s)]\nabla s \\
&\quad + A(f_i + h_{i_+})\phi_1(t) + B(f_i + h_{i_+})\phi_2(t) \\
&\leq \int_a^b G(s, s)[f_i(s, [u_1(s) - w_1(s)]^*, [u_2(s) - w_2(s)]^*) + h_{i_+}(s)]\nabla s \\
&\quad + A(f_i + h_{i_+})\phi_1(t) + B(f_i + h_{i_+})\phi_2(t) \\
&\leq \int_a^b G(s, s)[(d_1 d_2)^{\lambda_2 - \lambda_1} \|u_1\|^{\lambda_2} \|u_2\|^{\lambda_2} f_i(s, 1, 1) + h_{i_+}(s)]\nabla s \\
&\quad + A(f_i + h_{i_+})\phi_1(b) + B(f_i + h_{i_+})\phi_2(a) \\
&\leq ((d_1 d_2)^{\lambda_2 - \lambda_1} \|u_1\|^{\lambda_2} \|u_2\|^{\lambda_2} + 1) \int_a^b G(s, s)[f_i(s, 1, 1) + h_{i_+}(s)]\nabla s \\
&\quad + A((d_1 d_2)^{\lambda_2 - \lambda_1} \|u_1\|^{\lambda_2} \|u_2\|^{\lambda_2} + 1)\phi_1(b) \int_a^b G(s, s)[f_i(s, 1, 1) + h_{i_+}(s)]\nabla s \\
&\quad + B((d_1 d_2)^{\lambda_2 - \lambda_1} \|u_1\|^{\lambda_2} \|u_2\|^{\lambda_2} + 1)\phi_2(a) \int_a^b G(s, s)[f_i(s, 1, 1) + h_{i_+}(s)]\nabla s \\
&= ((d_1 d_2)^{\lambda_2 - \lambda_1} \|u_1\|^{\lambda_2} \|u_2\|^{\lambda_2} + 1)(1 + A\phi_1(b) + B\phi_2(a)) \int_a^b G(s, s)[f_i(s, 1, 1) + h_{i_+}(s)]\nabla s \\
&< \infty, \quad i = 1, 2.
\end{aligned}$$

Thus $F : P \to E$ is well defined. Now we shall prove that $F(P) \subseteq P$. For any $(u_1, u_2) \in P$, let $(v_1(t), v_2(t)) = F(u_1, u_2)(t)$. Then for $t \in [a, b]$, we get

$$v_i(t) = \int_a^b G(t,s)[f_i(s, [u_1(s) - w_1(s)]^*, [u_2(s) - w_2(s)]^*) + h_{i_+}(s)]\nabla s + A(f_i + h_{i_+})\phi_1(t) + B(f_i + h_{i_+})\phi_2(t)$$

$$\leq \int_a^b G(s,s)[f_i(s, [u_1(s) - w_1(s)]^*, [u_2(s) - w_2(s)]^*) + h_{i_+}(s)]\nabla s + A(f_i + h_{i_+})\phi_1(b) + B(f_i + h_{i_+})\phi_2(a)$$

and so

$$\|v_i\| \leq \int_a^b G(s,s)[f_i(s, [u_1(s) - w_1(s)]^*, [u_2(s) - w_2(s)]^*) + h_{i_+}(s)]\nabla s + A(f_i + h_{i_+})\phi_1(b) + B(f_i + h_{i_+})\phi_2(a),$$

For $t \in [a, b]$, the above relation and Lemma 3 gives:

$$v_i(t) = \int_a^b G(t,s)[f_i(s, [u_1(s) - w_1(s)]^*, [u_2(s) - w_2(s)]^*) + h_{i_+}(s)]\nabla s + A(f_i + h_{i_+})\phi_1(t) + B(f_i + h_{i_+})\phi_2(t)$$

$$\geq g(t) \int_a^b G(s,s)[f_i(s, [u_1(s) - w_1(s)]^*, [u_2(s) - w_2(s)]^*) + h_{i_+}(s)]\nabla s$$

$$+ A(f_i + h_{i_+})\frac{\phi_1(t)}{\phi_1(b)}\phi_1(b) + B(f_i + h_{i_+})\frac{\phi_2(t)}{\phi_2(a)}\phi_2(a)$$

$$\geq g(t) \int_a^b G(s,s)[f_i(s, [u_1(s) - w_1(s)]^*, [u_2(s) - w_2(s)]^*) + h_{i_+}(s)]\nabla s$$

$$+ A(f_i + h_{i_+})g(t)\phi_1(b) + B(f_i + h_{i_+})g(t)\phi_2(a)$$

$$= g(t)\left[\int_a^b G(s,s)[f_i(s, [u_1(s) - w_1(s)]^*, [u_2(s) - w_2(s)]^*) + h_{i_+}(s)]\nabla s \right.$$

$$\left. + A(f_i + h_{i_+})\phi_1(b) + B(f_i + h_{i_+})\phi_2(a)\right]$$

$$\geq g(t)\|v_i\|, \quad i = 1, 2.$$

This yields that $F(P) \subseteq P$.

Let $D \subset P$ be any bounded set. Then there exists a constant $M > 0$ such that $\|u_i\| \leq M$, $i = 1, 2$ for any $(u_1, u_2) \in D$. Furthermore for any $(u_1, u_2) \in D$ and $t \in [a, b]$, we find

$$0 \leq [u_i(t) - w_i(t)]^* \leq u_i(t) \leq \|u_i\| \leq M < M + 1, \quad i = 1, 2.$$

Thus, by Remark 2 and Lemma 8, for any $s \in [a, b]$, we have

$$f_i(s, [u_1(s) - w_1(s)]^*, [u_2(s) - w_2(s)]^*) + h_{i_+}(s) \leq f_i(s, M+1, M+1) + h_{i_+}(s)$$

$$\leq (M+1)^{2\lambda_1} f_i(s, 1, 1) + h_{i_+}(s) \leq ((M+1)^{2\lambda_1} + 1)[f_i(s, 1, 1) + h_{i_+}(s)], \quad i = 1, 2.$$

Consequently,

$$F_i(u_1, u_2)(t) = \int_a^b G(t,s)[f_i(s, [u_1(s) - w_1(s)]^*, [u_2(s) - w_2(s)]^*) + h_{i_+}(s)]\nabla s + A(f_i + h_{i_+})\phi_1(t) + B(f_i + h_{i_+})\phi_2(t)$$

$$\leq \int_a^b G(s,s)((M+1)^{2\lambda_1} + 1)[f_i(s, 1, 1) + h_{i_+}(s)]\nabla s$$

$$+ A\phi_1(b) \int_a^b G(s,s)((M+1)^{2\lambda_1} + 1)[f_i(s, 1, 1) + h_{i_+}(s)]\nabla s$$

$$+ B\phi_2(a) \int_a^b G(s,s)((M+1)^{2\lambda_1} + 1)[f_i(s, 1, 1) + h_{i_+}(s)]\nabla s$$

$$= ((M+1)^{2\lambda_1} + 1)(1 + A\phi_1(b) + B\phi_2(a)) \int_a^b G(s,s)[f_i(s, 1, 1) + h_{i_+}(s)]\nabla s$$

$$< \infty, \quad i = 1, 2.$$

Therefore $F(D)$ is uniformly bounded.

Similarly, we can easily find $F(D)$ is equicontinuous on $[a, b]$. Thus from the Ascoli-Arzela Theorem, we know that $F(D)$ is a relatively compact set.

Finally, from the continuity of $f_i, i = 1, 2$, it is not difficult to check that $F : P \to P$ is continuous. Hence $F : P \to P$ is a completely continuous operator. □

Theorem 2. *Let* $(H_1) - (H_7)$ *hold. For each r satisfying*

$$r > \max \left\{ 2C_1, 2C_2, ((r+1)^{2\lambda_1} + 1)(1 + A\phi_1(b) + B\phi_2(a)) \int_a^b G(s, s)[f_i(s, 1, 1) + h_{i_+}(s)]\nabla s \right\},$$

where $C_i (i = 1, 2)$ *are given in* (6). *The SSS* (1) *has at least one positive solution* $(\tilde{u}_1, \tilde{u}_2)$ *such that* $\|\tilde{u}_i\| > r, i = 1, 2.$

Proof. Assume that there exist $\lambda_0 \geq 1$ and $(\tilde{u}_1, \tilde{u}_2) \in \partial P_r$ such that $F(\tilde{u}_1, \tilde{u}_2) = \lambda_0(\tilde{u}_1, \tilde{u}_2)$ where $P_r = \{(u_1, u_2) \in P : \|u_1\| < r, \|u_2\| < r\}$. Then $\frac{1}{\lambda_0}(F_1(\tilde{u}_1, \tilde{u}_2), F_2(\tilde{u}_1, \tilde{u}_2)) = (\tilde{u}_1, \tilde{u}_2)$ and $0 < \frac{1}{\lambda_0} \leq 1$. Moreover for $t \in [a, b]$, we obtain

$$0 \leq [\tilde{u}_i(t) - w_i(t)]^* \leq \tilde{u}_i(t) \leq \|\tilde{u}_i\| = r < r + 1, \quad i = 1, 2,$$

from which, using Remark 2 and Lemma 8, for $t \in [a, b]$, we get

$$f_i(s, [u_1(s) - w_1(s)]^*, [u_2(s) - w_2(s)]^*) \leq f_i(s, r+1, r+1) \leq (r+1)^{2\lambda_1} f_i(s, 1, 1), \quad i = 1, 2.$$

Now, using Lemma 1 and Lemma 6 and the properties of the operators A, B, for $t \in [a, b]$, we get

$$\tilde{u}_i(t) = \frac{1}{\lambda_0} \left\{ \int_a^b G(t, s)[f_i(s, [\tilde{u}_1(s) - w_1(s)]^*, [\tilde{u}_2(s) - w_2(s)]^*) + h_{i_+}(s)]\nabla s \right.$$
$$\left. + A(f_i + h_{i_+})\phi_1(t) + B(f_i + h_{i_+})\phi_2(t) \right\}$$
$$\leq \int_a^b G(s, s)[f_i(s, [\tilde{u}_1(s) - w_1(s)]^*, [\tilde{u}_2(s) - w_2(s)]^*) + h_{i_+}(s)]\nabla s + A(f_i + h_{i_+})\phi_1(t) + B(f_i + h_{i_+})\phi_2(t)$$
$$\leq \int_a^b G(s, s)((r+1)^{2\lambda_1} + 1)[f_i(s, 1, 1) + h_{i_+}(s)]\nabla s$$
$$+ A\phi_1(b) \int_a^b G(s, s)((r+1)^{2\lambda_1} + 1)[f_i(s, 1, 1) + h_{i_+}(s)]\nabla s$$
$$+ B\phi_2(a) \int_a^b G(s, s)((r+1)^{2\lambda_1} + 1)[f_i(s, 1, 1) + h_{i_+}(s)]\nabla s$$
$$= ((r+1)^{2\lambda_1} + 1)(1 + A\phi_1(b) + B\phi_2(a)) \int_a^b G(s, s)[f_i(s, 1, 1) + h_{i_+}(s)]\nabla s.$$

Thus, we get

$$r \leq ((r+1)^{2\lambda_1} + 1)(1 + A\phi_1(b) + B\phi_2(a)) \int_a^b G(s, s)[f_i(s, 1, 1) + h_{i_+}(s)]\nabla s, \quad i = 1, 2.$$

This is a contradiction. Then by Theorem 1, we have

$$i(F, P_r, P) = 1. \tag{10}$$

On the other hand, let us choose the constant K such that

$$K > \left(\inf_{t \in [t_1, t_2]} g(t) \max_{t \in [a, b]} \int_{t_1}^{t_2} G(t, s)\nabla s \right)^{-1}.$$

In view of Lemma 8, there exists $N > 0$ such that

$$f_i(t, u_1, u_2) \geq K(u_1 + u_2), \quad u_1 \geq N, u_2 \geq N \quad and \quad t \in [t_1, t_2], \quad i = 1, 2.$$

Now, set

$$R = r + 2N(\inf_{t \in [t_1, t_2]} g(t))^{-1}. \tag{11}$$

Next, we show that $F(u_1, u_2) \not\leq (u_1, u_2)$ for any $(u_1, u_2) \in \partial P_R$. In fact, otherwise, there exists $(\hat{u_1}, \hat{u_2}) \in \partial P_R$ such that $(\hat{u_1}, \hat{u_2}) \geq F(\hat{u_1}, \hat{u_2}) = (F_1(\hat{u_1}, \hat{u_2}), F_2(\hat{u_1}, \hat{u_2}))$. From (6) and the fact that $(\hat{u_1}, \hat{u_2}) \in \partial P_R$, for $t \in [a, b]$, we get

$$w_i(t) \leq C_i g(t) \leq C_i \frac{\hat{u_i}(t)}{R} \quad i = 1, 2, \tag{12}$$

and noting that $R > r > \max\{2C_1, 2C_2\}$, from (11) and (12), for all $t \in [t_1, t_2]$, we obtain

$$\hat{u_i}(t) - w_i(t) \geq \left(1 - \frac{C_i}{R}\right)\hat{u_i}(t) \geq \frac{1}{2}\hat{u_i}(t) \geq \frac{1}{2}g(t)R \geq \frac{R}{2}\inf_{t \in [t_1, t_2]} g(t) > N > 0, \quad i = 1, 2.$$

Considering this, for $t \in [a, b]$, we get

$$\begin{aligned}
\hat{u_i}(t) &\geq \int_a^b G(t, s)[f_i(s, [\hat{u_1}(s) - w_1(s)]^*, [\hat{u_2}(s) - w_2(s)]^*) + h_{i_+}(s)]\nabla s \\
&\quad + A(f_i + h_{i_+})\phi_1(t) + B(f_i + h_{i_+})\phi_2(t) \\
&\geq \int_{t_1}^{t_2} G(t, s)f_i(s, [\hat{u_1}(s) - w_1(s)]^*, [\hat{u_2}(s) - w_2(s)]^*) \\
&= \int_{t_1}^{t_2} G(t, s)f_i(s, \hat{u_1}(s) - w_1(s), \hat{u_2}(s) - w_2(s))\nabla s \\
&\geq \int_{t_1}^{t_2} G(t, s)K(\hat{u_1}(s) - w_1(s) + \hat{u_2}(s) - w_2(s))\nabla s \\
&\geq \int_{t_1}^{t_2} G(t, s)KR \inf_{t \in [t_1, t_2]} g(t)\nabla s
\end{aligned}$$

and so

$$R \geq KR \inf_{t \in [t_1, t_2]} g(t) \max_{t \in [a, b]} \int_{t_1}^{t_2} G(t, s)\nabla s.$$

That is

$$K \leq \left(\inf_{t \in [t_1, t_2]} g(t) \max_{t \in [a, b]} \int_{t_1}^{t_2} G(t, s)\nabla s\right)^{-1}.$$

This contradicts the K that we choose. So from Theorem 1, we get

$$i(F, P_R, P) = 0. \tag{13}$$

Therefore, by equations (10) and (13), we have $i(F, P_R \backslash \overline{P_r}, P) = -1$. Then we see that the operator F has a fixed point $(\tilde{u_1}, \tilde{u_2})$ in P such that

$$r < \|\tilde{u_i}\| < R, \quad i = 1, 2.$$

Moreover, using this and inequality (6) for $t \in [a, b]$, we get

$$\tilde{u_i}(t) \geq g(t)\|\tilde{u_i}\| > rg(t) > 2C_i g(t) \geq 2w_i(t), \quad i = 1, 2.$$

Hence, $(\tilde{u_1}, \tilde{u_2})$ with $(w_1, w_2) \leq (\tilde{u_1}, \tilde{u_2})$ is a positive solution of the system (7). Therefore, by Lemma 7, $(\tilde{u_1} - w_1, \tilde{u_2} - w_2)$ is the positive solution of the SSS (1). $\square$

Example 1. Let $T = \{2^k : k \in Z\} \cup \{0\}$. Consider the following SSS,

$$\begin{cases} -u_i^{\triangle\nabla}(t) = f_i(t, u_1(t), u_2(t)) + h_i(t), & t \in (0, 1), \quad i = 1, 2, \\ u_i(0) - u_i^{\triangle}(0) = u_i(1) + u_i^{\triangle}(1) = 0, & i = 1, 2, \end{cases}$$

where

$$\begin{aligned}
f_1(t, u_1, u_2) &= t^2(1-t)u_1^{3/2}u_2^2 + \sqrt{u_1}, \quad h_1(t) = -t, \\
f_2(t, u_1, u_2) &= \frac{1}{10^3 t(1-t)}u_1^{3/2} + \frac{1}{10^2}\sqrt{u_1 + u_2}, \quad h_2(t) = -t^2.
\end{aligned}$$

Clearly f_1 and f_2 satisfy the condition (H_7). We can easily calculate the followings;

$$\int_0^1 G(s,s)h_{1_-}(s)\nabla s = \int_0^1 G(s,s)s\nabla s = \frac{1}{3}\int_0^1 (1+s)(2-s)s\nabla s = \frac{16}{35},$$

$$\int_0^1 G(s,s)h_{2_-}(s)\nabla s = \int_0^1 G(s,s)s^2\nabla s = \frac{1}{3}\int_0^1 (1+s)(2-s)s^2\nabla s = \frac{3776}{9765},$$

$$\int_0^1 h_{1_-}(s)\nabla s = \int_0^1 s\nabla s = \frac{2}{3}, \quad \int_0^1 h_{2_-}(s)\nabla s = \int_0^1 s^2\nabla s = \frac{4}{7},$$

$$\int_0^1 G(s,s)(1+h_{i_+}(s))\nabla s = \frac{1}{3}\int_0^1 (1+s)(2-s)\nabla s = \frac{44}{63} \quad for \quad i=1,2,$$

$$C_1 = \frac{1}{3}\int_0^1 h_{1_-}(s)\nabla s\phi_1(1)\phi_2(0) = \frac{8}{9}, \quad C_2 = \frac{1}{3}\int_0^1 h_{2_-}(s)\nabla s\phi_1(1)\phi_2(0) = \frac{16}{21},$$

And for $(t,u_1,u_2) \in [0,1] \times [0,r] \times [0,r]$

$$K_1 = \max\left\{t^2(1-t)u_1^{3/2}u_2^2 + \sqrt{u_1} + 1)\right\} = \frac{1}{12}r^{7/2} + \sqrt{r} + 1,$$

$$K_2 = \max\left\{\frac{1}{10^3t(1-t)}u_1^{3/2} + \frac{1}{10^2}\sqrt{u_1+u_2} + 1\right\} = \frac{1}{250}r^{3/2} + \frac{1}{10^2}\sqrt{2r} + 1.$$

If we choose $r = \dfrac{17}{9}$, we have

$$r > \max\left\{\frac{16}{9}, \frac{32}{21}, K_1\frac{44}{63}, K_2\frac{44}{63}\right\}.$$

Then, by Theorem 2, the dynamic system has two positive solutions $(\tilde{u}_1, \tilde{u}_2)$ and $(\hat{u}_1, \hat{u}_2)$ such that

$$0 < \|\hat{u}_i\| < \frac{17}{9} < \|\tilde{u}_i\|, \quad i=1,2.$$

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