Physical Science International Journal

20(3): 1-5, 2018; Article no.PSIJ.45915
ISSN: 2348-0130

Determination of Energy Levels of a Quantum Bouncing Ball with the Numerical Support of Maple Computer Program

E. Omugbe[1*]

[1]*Department of Physics, Nigeria Maritime University, Okerenkoko, Warri, Delta State, Nigeria.*

Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/PSIJ/2018/45915
Editor(s):
(1) Dr. Mohd Rafatullah, Division of Environmental Technology, School of Industrial Technology, Universiti Sains Malaysia, Malaysia.
(2) Dr. Roberto Oscar Aquilano, School of Exact Science, National University of Rosario (UNR), Rosario, Physics Institute (IFIR)(CONICET-UNR), Argentina.
Reviewers:
(1) Andrej B. Arbuzov, Joint Institute for Nuclear Research, Russia.
(2) Omenyi Louis, Alex Ekwueme Federal University, Nigeria.
(3) Adel H. Phillips, Ain-Shams University, Egypt.
(4) Victor Christianto, Malang Institute of Agriculture, Indonesia.
(5) Pasupuleti Venkata Siva Kumar, VNR Vignana Jyothi Institute of Engineering and Technology, India.
Complete Peer review History: http://www.sciencedomain.org/review-history/27890

Mini-review Article

Received 05 October 2018
Accepted 10 December 2018
Published 20 December 2018

ABSTRACT

The dynamics of the quantum bouncing ball under the influence of gravity has been studied. The Fourier transform techniques was first used to derive the wave function (Airy function) of the time-independent Schrodinger wave equation for a linear potential in the form of the well-known Airy integral. The asymptotic dependence of the Airy function was presented. In order to obtain the numerical solution of the energy levels of the "bouncing ball", the power series method was first used to derive the Airy function in power series form. Subsequently, the energy levels were then computed with the support of Maple Software. The results are tabulated with the WKB (Wentzel, Kramers, and Brillouin) approximation calculations.

Keywords: Fourier transform; Airy equation; quantum bouncing ball; WKB method; Schrodinger wave equation.

**Corresponding author: E-mail: omugbeekwevugbe@gmail.com;*

1. INTRODUCTION

The solution to the time-independent Schrodinger wave equation for a particle bouncing on a perfectly reflecting hard surface under the influence of the Earth gravitational potential energy ($V(x) = mgx(x > 0)$) is presented in this paper. This linear potential has other practical implications in the study of quark-antiquark energy spectrum [1] and also in the study of the condensation behaviour of a Bose-Einstein idea gas [2]. Nesvizhevsky et al. [3], have shown experimentally that quantum effect can be observed macroscopically by using a very small mass such as an ultra-cold neutron bouncing on a perfectly reflecting mirror. Also, Jenke et al. [4] have performed gravity experiments with ultra-cold neutrons to calculate the energy levels using a Gravity Resonance Spectroscopy (GRS). The dynamics of the quantum bouncing ball have been discussed in a number of pedagogical texts such as [1,5-7]. This paper relies on the brief discussion of the “bouncing ball” in [5, p. 26 – 29]. It is worthwhile to mention also that the calculations of the energy levels in the pedagogical textbooks cited herein were based on the WKB approximation calculations and their asymptotic dependence. Following the omission on the numerical solution, a tractable approach namely the power series method with the support of Maple computer program was used to compute the energy levels.

2. SCHRODINGER WAVE EQUATION FOR A LINEAR POTENTIAL

The time-independent Schrodinger wave equation is given as

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + V(x)u = Eu \ . \tag{1}$$

Where

$$V(x) = \begin{cases} mgx, & x > 0 \\ \infty, & x < 0 \end{cases} \ . \tag{2}$$

Setting $x = az + b$, with $a = \left(\frac{\hbar^2}{2m^2 g}\right)^{\frac{1}{3}}$ and $b = \frac{E}{mg}$, (1) transforms as follows;

$$\frac{du}{dz} = \frac{dx}{dz}\frac{du}{dx} \tag{3}$$

The second derivative of (3) gives

$$\frac{d^2u}{dz^2} = \left(\frac{dx}{dz}\right)^2 \frac{d^2u}{dx^2} + \frac{du}{dx}\frac{d^2x}{dz^2} = a^2\frac{d^2u}{dx^2} \ . \tag{4}$$

On substituting (4) into (1) with x, a and b defined above, (1) reduces to

$$\frac{d^2u}{dz^2} - zu = 0 \ . \tag{5}$$

Equation (5) is the well know Airy differential equation which has practical applications in physics (see [8]). The solution to (5) yields the Airy functions $Ai(z)$ and $Bi(z)$ of which $Bi(z)$ blows up as its argument grows and is not accepted as a wave function [9].

3. ASYMPTOTIC DEPENDENCE OF THE AIRY FUNCTION $Ai(z)$

Fourier transform properties were first used to transform (5) into the Airy Integral. Let

$$u^n(z) = (ik)^n \tilde{u}(k) \tag{6}$$

$$z^n u(z) = (i)^n \tilde{u}^n(k) \tag{7}$$

$$u(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{u}(k)\exp(-ikz)dk \ . \tag{8}$$

Where $u(z)$ is the function and $\tilde{u}(k)$ is the transform. Substituting (6) and (7) for $n = 2$ and $n = 1$ respectively into (5) yields

$$-k^2\tilde{u}(k) - i\frac{d\tilde{u}(k)}{dk} = 0 \ . \tag{9}$$

Clearly, (9) is a first order ordinary differential equation which can be evaluated by the method of separation of variables. The solution of (9) is given as

$$\tilde{u}(k) = expi(\frac{k^3}{3}). \tag{10}$$

Substituting (10) into (8) gives

$$u(z) \equiv Ai(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} expi\left(\frac{k^3}{3} + kz\right)dk \ . \tag{11}$$

The improper integral in (11) is the solution to (5) and can be evaluated using the method of steepest descents discussed in [10, p. 90 – 94] and [11]. This method gives the leading term of the asymptotic behaviour of $Ai(z)$ as

$$Ai(z) \approx \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} \exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right) . \tag{12}$$

The other correction terms in (12) can be obtained with the formula given in [12, p. 448]

$$Ai(z) \approx \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} \exp(-\zeta) \sum_0^{\infty} (-1)^k c_k \zeta^{-k}$$
for large z . (13)

Where $\zeta = \frac{2}{3} z^{\frac{3}{2}}$ and

$$c_k = \frac{\Gamma\left(3k+\frac{1}{2}\right)}{54^k k!\, \Gamma\left(k+\frac{1}{2}\right)}, \qquad c_0 = 1$$

4. DETERMINATION OF THE ENERGY LEVELS OF THE BOUNCING BALL

The energy levels are the zeros of Airy function $Ai(z)$ in the units of $\left(\frac{mg^2\hbar^2}{2}\right)^{\frac{1}{3}}$. In order to find the zeros, the power series techniques of differential equation is applied to (5).

Let the solution to (5) be $u(z) = \sum_{n=0}^{\infty} d_n z^n$. (14)

Putting (14) into (5) yields the recurrent relation

$$d_n = \frac{d_{n-3}}{(n-1)n} \qquad \text{For } n \geq 3 \text{ with } d_2 = 0 . \quad (15)$$

Substituting the constant terms in (15) into (14) allows us to write the solution as a linear combination of two linearly independent functions.

$$u(z) = d_0 u_1(z) + d_1 u_2(z) . \quad (16)$$

Where

$$u_1(z) = 1 + \frac{z^3}{2(3)} + \frac{z^6}{2(3)(5)(6)} + \cdots \quad (17)$$

$$u_2(z) = z + \frac{z^4}{(3)(4)} + \frac{z^7}{(3)(4)(6)(7)} + \cdots \quad (18)$$

Using the initial value conditions

$$u(0) = \frac{1}{(3)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}, \quad \frac{du(0)}{dz} = 0 \quad (19)$$

$$\frac{du(0)}{dz} = -\frac{1}{(3)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)}, \quad u(0) = 0 . \quad (20)$$

We can write (16) as

$$u(z) \equiv Ai(z) = \frac{1}{(3)^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)} u_1(z) - \frac{1}{(3)^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)} u_2(z) . \quad (21)$$

Equation (21) is the power series form of the Airy function $Ai(z)$ which oscillates for negative values of its argument but collapses to zeros for positive values.

Gea-Banacloche [9] states that the exact analytical solution of the zeros of (21) are not available. However, an approximation solution such as the Wentzel, Kramers, and Brillouin (WKB) method has been used to find the zeros (see [1] and problem 8.6 in [7]). The zeros asymptotic dependence are given as

$$\frac{E_n}{\left(\frac{mg^2\hbar^2}{2}\right)^{\frac{1}{3}}} \approx \left[\frac{3\pi}{2}\left(n - \frac{1}{4}\right)\right]^{\frac{2}{3}} . \quad (22)$$

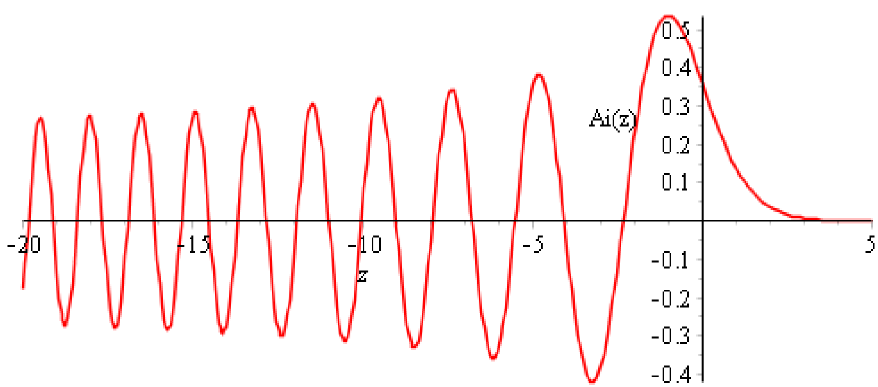

Fig. 1. This shows the graph of $Ai(z)$ function

To find numerical solution of (21), the two linearly independent functions $u_1(z)$ and $u_2(z)$ are expanded in Maple to $O(z^{150})$ as shown below.

$$ode1 := diff(u1(z), z, z) - z*u1(z) = 0; Order := 150;$$
$$dsolve\left(\left\{ode1, u1(0) = \frac{1}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}, \mathrm{D}(u1)(0) = 0\right\}, u1(z), type = series\right);$$
$$ode2 := diff(u2(z), z, z) - z*u2(z) = 0;$$
$$dsolve\left(\left\{ode2, u2(0) = 0, \mathrm{D}(u2)(0) = -\frac{1}{3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)}\right\}, u2(z), type = series\right);$$

Using the boundary condition $x = 0$ such that $u\left(\frac{x-b}{a}\right) \equiv Ai\left(-\frac{b}{a}\right) = 0$ helps us to find the roots.

The results are tabulated below:

Table 1. This shows the first seven energy levels in units of $\left(\frac{mg^2\hbar^2}{2}\right)^{\frac{1}{3}}$

n	Numerical Solution	WKB Approximation
1	2.338107410	2.320250794
2	4.087949446	4.081810015
3	5.520559833	5.517163872
4	6.786708296	6.784454481
5	7.944095215	7.942486664
6	9.024004490	9.021373236
7	10.01217466	10.03914214

5. CONCLUSION

The dynamics of a quantum bouncing ball under the influence of Earth's gravity, has been studied with a view to determining the energy levels arising from the solution to the time-independent Schrodinger wave equation for a linear potential. The power series method was used to solve (5) and yields the wave function (Airy function) which has many practical applications in physics and applied mathematics. The zeros of the Airy function were then used to find the energy levels which enables us to determine the allowed heights with which a quantum particle would bounce.

COMPETING INTERESTS

Author has declared that no competing interests exist.

REFERENCES

1. Sakurai JJ, Napolitano J. Modern quantum mechanics. 2nd Ed. Addison-Wesley. Boston; 2011.
2. Berberan-Santos MN, et al. Classical and quantum study of the motion of a particle in a gravitational field. Journal of Mathematical Chemistry. 2005;37(2):101-115.
3. Nesvizhevsky VV, et al. Study of the Neutron quantum states in the gravity field. European Physical Journal C. 2005;40: 479-91.
4. Jenke et al. arxiv.1510.03078; 2015.
5. Davies PCW, Betts DS. Quantum Mechanics. 2nd Ed. Chapman and Hall, London; 1994.
6. Merzbacher E. Quantum mechanics, 3rd ed. John Wiley and Sons Inc. New York; 1998.
7. Griffiths DJ. Introduction to quantum mechanics. Prentice Hall Inc. Upper Saddle River, NJ USA; 1995.
8. Vallee O, Soares M. Airy functions and applications to physics. Imperial College Press. London; 2004.
9. Gea-Banacloche J. A quantum bouncing ball. Am. J. Phys. 1999;67(9):776-782.

10. Wong R. Asymptotic approximations of integrals. Society for Industrial and Applied Mathematics, Philadelphia; 2001.
11. Navarro FG, Campos Garcia JC, Molinar Taberes ME. Approximate solutions of the airy equation with numerical support of MATLAB. IJSIMR. 2015;3(6):63-72.
12. Abramowitz M, Stegun IA. Handbook of Mathematical Functions with formulas, graphs and mathematical tables. Nat. Bur. Standards, Appli. Math. Series 55, US Government Printing Office Washington DC; 1964.

Peer-review history:
The peer review history for this paper can be accessed here:
http://www.sciencedomain.org/review-history/27890