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Summation Formulas for Generalized Tetranacci Numbers

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, closed forms of the summation formulas for generalized Tetranacci numbers are presented. Then, some previous results are recovered as particular cases of the present results. As special cases, we give summation formulas of Tetranacci, Tetranacci-Lucas, fourth order Pell, fourth order Pell-Lucas, fourth order Jacobsthal, fourth order Jacobsthal-Lucas numbers.

Keywords: Tetranacci numbers; Tetranacci-Lucas numbers; sum formulas; summing formulas.

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1 INTRODUCTION

There have been so many studies of the sequences of numbers in the literature which are defined recursively. Two of these type of sequences are the sequences of Tetranacci and Tetranacci-Lucas which are special case of generalized Tetranacci numbers. A generalized Tetranacci sequence:

$$\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3; r, s, t, u)\}_{n\geq 0}$$

is defined by the fourth-order recurrence relations

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \quad (1.1)$$

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with the initial values W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers not all being zero and r, s, t, u are real numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [1,2,3,4,5,6].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1, 2, 3, ...$. Therefore, recurrence (1.1) holds for all integer n.

For some specific values of W_0, W_1, W_2, W_3 and r, s, t, u, it is worth presenting these special Tetranacci numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s, t, u and initial values.

The first few values of the sequences with non-negative indices are shown below (see Table 2).

The first few values of the sequences with negative indices are presented in the following table (Table 3).

For easy writing, from now on, we drop the superscripts from the sequences, for example we write P_n for $P_n^{(4)}$.

In this work, we investigate linear summation formulas of generalized Tetranacci and Gaussian genera-lized Tetranacci numbers. Some summing formulas of the Pell and Pell-Lucas numbers are well known and given in [7, 8], see also [9]. For linear sums of Fibonacci, Tribonacci, Tetranacci, Pentanacci and Hexanacci numbers, see [10,11], [12,13,14], [15, 5], [16], and [17] respectively.

Table 1. A few special case of generalized Tetranacci sequences

Sequences (Numbers)	Notation	OEIS [18]
Tetranacci	$\{M_n\} = \{W_n(0,1,1,2;1,1,1,1)\}$	A000078
Tetranacci-Lucas	$\{R_n\} = \{W_n(4,1,3,7;1,1,1,1)\}$	A073817
fourth order Pell	$\{P_n^{(4)}\} = \{W_n(0,1,2,5;2,1,1,1)\}$	A103142
fourth order Pell-Lucas	$\{Q_n^{(4)}\} = \{W_n(4,2,6,17;2,1,1,1)\}$	-
fourth order Jacobsthal	$\{J_n^{(4)}\} = \{W_n(0,1,1,1;1,1,1,2)\}$	-
fourth order Jacobsthal-Lucas	$\{j_n^{(4)}\} = \{W_n(2,1,5,10;1,1,1,2)\}$	A226309

Table 2. A few values of the sequences with positive subscript

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
M_n	0	1	1	2	4	8	15	29	56	108	208	401	773	1490
R_n	4	1	3	7	15	26	51	99	191	367	708	1365	2631	5071
$P_n^{(4)}$	0	1	2	5	13	34	88	228	591	1532	3971	10293	26680	69156
$Q_n^{(4)}$	4	2	6	17	46	117	303	786	2038	5282	13691	35488	91987	238435
$J_n^{(4)}$	0	1	1	1	3	7	13	25	51	103	205	409	819	1639
$j_n^{(4)}$	2	1	5	10	20	37	77	154	308	613	1229	2458	4916	9829

Table 3. A few values of the sequences with negative subscript

n	1	2	3	4	5	6	7	8	9	10	11	12	13
M_{-n}	0	0	1	-1	0	0	2	-3	1	0	4	-8	5
R_{-n}	-1	-1	-1	7	-6	-1	-1	15	-19	4	-1	31	-53
$P_{-n}^{(4)}$	0	0	1	-1	0	-1	4	-4	2	-7	17	-18	17
$Q_{-n}^{(4)}$	-1	-1	-4	11	-6	2	-22	43	-31	34	-111	182	-170
$J_{-n}^{(4)}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{5}{8}$	$-\frac{3}{16}$	$-\frac{19}{32}$	$\frac{13}{64}$	$\frac{77}{128}$	$-\frac{51}{256}$	$-\frac{307}{512}$	$\frac{205}{1024}$	$\frac{1229}{2048}$	$-\frac{819}{4096}$	$-\frac{4915}{8192}$
$j_{-n}^{(4)}$	1	$\frac{1}{2}$	$-\frac{5}{4}$	$\frac{7}{8}$	$\frac{7}{16}$	$\frac{7}{32}$	$-\frac{89}{64}$	$\frac{103}{128}$	$\frac{103}{256}$	$\frac{103}{512}$	$-\frac{1433}{1024}$	$\frac{1639}{2048}$	$\frac{1639}{4096}$

2 LINEAR SUM FORMULAS OF GENERALIZED TETRANACCI NUMBERS WITH POSITIVE SUBSCRIPTS

The following theorem presents some linear summing formulas of generalized Tetranacci numbers with positive subscripts.

Theorem 2.1. For $n \geq 0$ we have the following formulas:

(a) (Sum of the generalized Tetranacci numbers) If $r+s+t+u-1 \neq 0$, then

$$\sum_{k=0}^{n} W_k = \frac{W_{n+4}+(1-r)W_{n+3}+(1-r-s)W_{n+2}+(1-r-s-t)W_{n+1}+K_1}{r+s+t+u-1}.$$

where

$$K_1 = -W_3+(r-1)W_2+(r+s-1)W_1+(r+s+t-1)W_0.$$

(b) If $(r+s+t+u-1)(r-s+t-u+1) \neq 0$ then

$$\sum_{k=0}^{n} W_{2k} = \frac{(1-s-u)W_{2n+2}+(t+rs+ru)W_{2n+1}+(t^2-u^2+rt-su+u)W_{2n}+(ru+tu)W_{2n-1}+K_2}{(r-s+t-u+1)(r+s+t+u-1)}$$

where

$$K_2 = -(r+t)W_3+(s+u+rt+r^2-1)W_2+(st-ru-t)W_1+(r^2-s^2+t^2+2s+u+2rt-su-1)W_0.$$

and

$$\sum_{k=0}^{n} W_{2k+1} = \frac{(r+t)W_{2n+2}+(-s^2+t^2-u^2+rt-2su+s+u)W_{2n+1}+(t+ru-st)W_{2n}-u(s+u-1)W_{2n-1}+K_3}{(r+s+t+u-1)(r-s+t-u+1)}.$$

where

$$K_3 = (s+u-1)W_3-(t+rs+ru)W_2+(r^2-s^2+rt-su+2s+u-1)W_1-u(r+t)W_0.$$

(c) If $r+t \neq 0 \wedge s+u-1 = 0$ then

$$\sum_{k=0}^{n} W_{2k} = \frac{W_{2n+1}+tW_{2n}+uW_{2n-1}-W_3+rW_2-uW_1+(r+t)W_0}{r+t}$$

and

$$\sum_{k=0}^{n} W_{2k+1} = \frac{W_{2n+2}+tW_{2n+1}+uW_{2n}-W_2+rW_1-uW_0}{r+t}.$$

Note that (c) is a special case of (b).

Proof.

(a) Using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}$$

i.e.

$$uW_{n-4} = W_n-rW_{n-1}-sW_{n-2}-tW_{n-3}$$

we obtain

$$\begin{aligned}
uW_0 &= W_4 - rW_3 - sW_2 - tW_1 \\
uW_1 &= W_5 - rW_4 - sW_3 - tW_2 \\
uW_2 &= W_6 - rW_5 - sW_4 - tW_3 \\
uW_3 &= W_7 - rW_6 - sW_5 - tW_4 \\
&\vdots \\
uW_{n-4} &= W_n - rW_{n-1} - sW_{n-2} - tW_{n-3} \\
uW_{n-3} &= W_{n+1} - rW_n - sW_{n-1} - tW_{n-2} \\
uW_{n-2} &= W_{n+2} - rW_{n+1} - sW_n - tW_{n-1} \\
uW_{n-1} &= W_{n+3} - rW_{n+2} - sW_{n+1} - tW_n \\
uW_n &= W_{n+4} - rW_{n+3} - sW_{n+2} - tW_{n+1}
\end{aligned}$$

If we add the above equations by side by, we get (a).

(b) and (c) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4}$$

we obtain

$$\begin{aligned}
rW_3 &= W_4 - sW_2 - tW_1 - uW_0 \\
rW_5 &= W_6 - sW_4 - tW_3 - uW_2 \\
rW_7 &= W_8 - sW_6 - tW_5 - uW_4 \\
rW_9 &= W_{10} - sW_8 - tW_7 - uW_6 \\
&\vdots \\
rW_{2n-1} &= W_{2n} - sW_{2n-2} - tW_{2n-3} - uW_{2n-4} \\
rW_{2n+1} &= W_{2n+2} - sW_{2n} - tW_{2n-1} - uW_{2n-2} \\
rW_{2n+3} &= W_{2n+4} - sW_{2n+2} - tW_{2n+1} - uW_{2n}
\end{aligned}$$

Now, if we add the above equations by side by, we get

$$\begin{aligned}
r(-W_1 + \sum_{k=0}^{n} W_{2k+1}) &= (W_{2n+2} - W_2 - W_0 + \sum_{k=0}^{n} W_{2k}) - s(-W_0 + \sum_{k=0}^{n} W_{2k}) \qquad (2.1) \\
&\quad -t(-W_{2n+1} + \sum_{k=0}^{n} W_{2k+1}) - u(-W_{2n} + \sum_{k=0}^{n} W_{2k}).
\end{aligned}$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3} - uW_{n-4}$$

we write the following obvious equations;

$$\begin{aligned} rW_2 &= W_3 - sW_1 - tW_0 - uW_{-1} \\ rW_4 &= W_5 - sW_3 - tW_2 - uW_1 \\ rW_6 &= W_7 - sW_5 - tW_4 - uW_3 \\ rW_8 &= W_9 - sW_7 - tW_6 - uW_5 \\ &\vdots \\ rW_{2n-2} &= W_{2n-1} - sW_{2n-3} - tW_{2n-4} - uW_{2n-5} \\ rW_{2n} &= W_{2n+1} - sW_{2n-1} - tW_{2n-2} - uW_{2n-3} \\ rW_{2n+2} &= W_{2n+3} - sW_{2n+1} - tW_{2n} - uW_{2n-1} \\ rW_{2n+4} &= W_{2n+5} - sW_{2n+3} - tW_{2n+2} - uW_{2n+1} \end{aligned}$$

Now, if we add the above equations by side by, we obtain

$$\begin{aligned} r(-W_0 + \sum_{k=0}^{n} W_{2k}) &= (-W_1 + \sum_{k=0}^{n} W_{2k+1}) - s(-W_{2n+1} + \sum_{k=0}^{n} W_{2k+1}) - t(-W_{2n} + \sum_{k=0}^{n} W_{2k}) \\ &\quad -u(-W_{2n+1} - W_{2n-1} + W_{-1} + \sum_{k=0}^{n} W_{2k+1}). \end{aligned}$$

Since

$$W_{-1} = -\frac{t}{u}W_0 - \frac{s}{u}W_1 - \frac{r}{u}W_2 + \frac{1}{u}W_3$$

we have

$$\begin{aligned} r(-W_0 + \sum_{k=0}^{n} W_{2k}) &= (-W_1 + \sum_{k=0}^{n} W_{2k+1}) - s(-W_{2n+1} + \sum_{k=0}^{n} W_{2k+1}) - t(-W_{2n} + \sum_{k=0}^{n} W_{2k}) \\ &\quad -u(-W_{2n+1} - W_{2n-1} + (-\frac{t}{u}W_0 - \frac{s}{u}W_1 - \frac{r}{u}W_2 + \frac{1}{u}W_3) + \sum_{k=0}^{n} W_{2k+1}). \end{aligned} \tag{2.2}$$

Then, solving the system (2.1)-(2.2), the required results of (b) and (c) follow.

Taking $r = s = t = u = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following proposition.

Proposition 2.2. If $r = s = t = u = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} W_k = \frac{1}{3}(W_{n+4} - W_{n+2} - 2W_{n+1} - W_3 + W_1 + 2W_0)$.

(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{3}(-W_{2n+2} + 3W_{2n+1} + W_{2n} + 2W_{2n-1} - 2W_3 + 3W_2 - W_1 + 4W_0)$.

(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{3}(2W_{2n+2} + W_{2n} - W_{2n-1} + W_3 - 3W_2 + 2W_1 - 2W_0)$.

Proof. We take $r = s = t = u = 1$ in Theorem 2.1 (a) and (b). Note that in this case we have

$$\begin{aligned} r+s+t+u-1 &= 3, \\ (r+s+t+u-1)(r-s+t-u+1) &= 3. \end{aligned}$$

(a)

$$\sum_{k=0}^{n} W_k = \frac{W_{n+4} + 0 \times W_{n+3} + (-1)W_{n+2} + (-2)W_{n+1} + K_1}{3}$$

where

$$K_1 = -W_3 + 0 \times W_2 + 1 \times W_1 + 2 \times W_0.$$

(b)

$$\sum_{k=0}^{n} W_{2k} = \frac{(-1)W_{2n+2} + 3 \times W_{2n+1} + 1 \times W_{2n} + 2 \times W_{2n-1} + K_2}{3}$$

where

$$K_2 = -2 \times W_3 + 3 \times W_2 + (-1)\, W_1 + 4 \times W_0.$$

(c)

$$\sum_{k=0}^{n} W_{2k+1} = \frac{2 \times W_{2n+2} + 0 \times W_{2n+1} + 1 \times W_{2n} - 1 \times W_{2n-1} + K_3}{(r+s+t+u-1)\,(r-s+t-u+1)}$$

where

$$K_3 = 1 \times W_3 - 3 \times W_2 + 2 \times W_1 - 2 \times W_0.$$

From the above proposition, we have the following corollary which gives linear sum formulas of Tetranacci numbers (take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2$).

Corollary 2.3. For $n \geq 0,$ Tetranacci numbers have the following properties.

(a) $\sum_{k=0}^{n} M_k = \frac{1}{3}(M_{n+4} - M_{n+2} - 2M_{n+1} - 1).$

(b) $\sum_{k=0}^{n} M_{2k} = \frac{1}{3}(-M_{2n+2} + 3M_{2n+1} + M_{2n} + 2M_{2n-1} - 2).$

(c) $\sum_{k=0}^{n} M_{2k+1} = \frac{1}{3}(2M_{2n+2} + M_{2n} - M_{2n-1} + 1).$

Proof. We take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2.$

(a)

$$K_1 = -M_3 + M_1 + 2M_0 = -1.$$

(b)

$$K_2 = -2M_3 + 3M_2 - M_1 + 4M_0 = -2.$$

(c)

$$K_3 = M_3 - 3M_2 + 2M_1 - 2M_0 = 1.$$

Taking $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$ in the above proposition, we have the following corollary which presents linear sum formulas of Tetranacci-Lucas numbers.

Corollary 2.4. For $n \geq 0,$ Tetranacci-Lucas numbers have the following properties.

(a) $\sum_{k=0}^{n} R_k = \frac{1}{3}(R_{n+4} - R_{n+2} - 2R_{n+1} + 2).$

(b) $\sum_{k=0}^{n} R_{2k} = \frac{1}{3}(-R_{2n+2} + 3R_{2n+1} + R_{2n} + 2R_{2n-1} + 10).$

(c) $\sum_{k=0}^{n} R_{2k+1} = \frac{1}{3}(2R_{2n+2} + R_{2n} - R_{2n-1} - 8).$

Proof. We take $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7.$

(a)

$$K_1 = -R_3 + R_1 + 2R_0 = 2.$$

(b)

$$K_2 = -2R_3 + 3R_2 - R_1 + 4R_0 = 10.$$

(c)

$$K_3 = R_3 - 3R_2 + 2R_1 - 2R_0 = -8.$$

Taking $r = 2, s = t = u = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following proposition.

Proposition 2.5. If $r = 2, s = t = u = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} W_k = \frac{1}{4}(W_{n+4} - W_{n+3} - 2W_{n+2} - 3W_{n+1} - W_3 + W_2 + 2W_1 + 3W_0).$

(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{8}(-W_{2n+2} + 5W_{2n+1} + 2W_{2n} + 3W_{2n-1} - 3W_3 + 7W_2 - 2W_1 + 9W_0)$.

(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{8}(3W_{2n+2} + W_{2n+1} + 2W_{2n} - W_{2n-1} + W_3 - 5W_2 + 6W_1 - 3W_0)$.

From the last proposition, we have the following corollary which gives linear sum formulas of fourth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5$).

Corollary 2.6. For $n \geq 0$, fourth-order Pell numbers have the following properties:

(a) $\sum_{k=0}^{n} P_k = \frac{1}{4}(P_{n+4} - P_{n+3} - 2P_{n+2} - 3P_{n+1} - 1)$.

(b) $\sum_{k=0}^{n} P_{2k} = \frac{1}{8}(-P_{2n+2} + 5P_{2n+1} + 2P_{2n} + 3P_{2n-1} - 3)$.

(c) $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{8}(3P_{2n+2} + P_{2n+1} + 2P_{2n} - P_{2n-1} + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 4, Q_1 = 2, Q_2 = 6, Q_3 = 17$ in the last proposition, we have the following corollary which presents linear sum formulas of fourth-order Pell-Lucas numbers.

Corollary 2.7. For $n \geq 0$, fourth-order Pell-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} Q_k = \frac{1}{4}(Q_{n+4} - Q_{n+3} - 2Q_{n+2} - 3Q_{n+1} + 5)$.

(b) $\sum_{k=0}^{n} Q_{2k} = \frac{1}{8}(-Q_{2n+2} + 5Q_{2n+1} + 2Q_{2n} + 3Q_{2n-1} + 23)$.

(c) $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{8}(3Q_{2n+2} + Q_{2n+1} + 2Q_{2n} - Q_{2n-1} - 13)$.

If $r = 1, s = 1, t = 1, u = 2$ then $(r + s + t + u - 1)(r - s + t - u + 1) = 0$ so we can't use Theorem 2.1 (b). In other words, the method of the proof Theorem 2.1 (b) can't be used to find $\sum_{k=0}^{n} W_{2k}$ and $\sum_{k=0}^{n} W_{2k+1}$.

Proposition 2.8. If $r = 1, s = 1, t = 1, u = 2$ then for $n \geq 0$ we have the following formula:

$$\sum_{k=0}^{n} W_k = \frac{1}{4}(W_{n+4} - W_{n+2} - 2W_{n+1} - W_3 + W_1 + 2W_0).$$

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1$ in the last proposition, we have the following corollary which presents linear sum formula of fourth-order Jacobsthal numbers.

Corollary 2.9. For $n \geq 0$, fourth order Jacobsthal numbers have the following property:

$$\sum_{k=0}^{n} J_k = \frac{1}{4}(J_{n+4} - J_{n+2} - 2J_{n+1} - J_3 + J_1 + 2J_0).$$

From the last proposition, we have the following corollary which gives linear sum formula of fourth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10$).

Corollary 2.10. For $n \geq 0$, fourth order Jacobsthal-Lucas numbers have the following property:

$$\sum_{k=0}^{n} j_k = \frac{1}{4}(j_{n+4} - j_{n+2} - 2j_{n+1} - 5).$$

3 LINEAR SUM FORMULAS OF GENERALIZED TETRANACCI NUMBERS WITH NEGATIVE SUBSCRIPTS

The following theorem present some linear summing formulas of generalized Tetranacci numbers with negative subscripts.

Theorem 3.1. For $n \geq 1$ we have the following formulas:

(a) (Sum of the generalized Tetranacci numbers with negative indices) If $r+s+t+u-1 \neq 0,$ then

$$\sum_{k=1}^{n} W_{-k} = \frac{-W_{-n+3}+(r-1)W_{-n+2}+(r+s-1)W_{-n+1}+(r+s+t-1)W_{-n}+K_4}{r+s+t+u-1}$$

where

$$K_4 = W_3+(1-r)W_2+(1-r-s)W_1+(1-s-r-t)W_0.$$

(b) If $(r-s+t-u+1)\,(r+s+t+u-1) \neq 0$ then

$$\sum_{k=1}^{n} W_{-2k} = \frac{(s+u-1)W_{-2n+2}-(t+rs+ru)\,W_{-2n+1}+(r^2-s^2+rt-su+2s+u-1)W_{-2n}-u\,(r+t)\,W_{-2n-1}+K_5}{(r-s+t-u+1)\,(r+s+t+u-1)}$$

where

$$K_5 = (r+t)W_3+(1-r^2-rt-s-u)W_2+(t+ru-st)\,W_1+(1-r^2+s^2-t^2-2rt+su-2s-u)W_0.$$

and

$$\sum_{k=1}^{n} W_{-2k+1} = \frac{-(r+t)W_{-2n+2}+(r^2+rt+s+u-1)W_{-2n+1}+(st-t-ru)W_{-2n}+(u^2+su-u)W_{-2n-1}+K_6}{(r-s+t-u+1)\,(r+s+t+u-1)}$$

where

$$K_6 = (1-s-u)W_3+(t+ru+rs)W_2+(1-r^2+s^2-rt+su-2s-u)W_1+u(r+t)W_0.$$

(c) If $r+t \neq 0 \wedge s+u-1=0$ then

$$\sum_{k=1}^{n} W_{-2k} = \frac{-W_{-2n+1}+rW_{-2n}-uW_{-2n-1}+W_3-rW_2+uW_1-(r+t)W_0}{r+t}$$

and

$$\sum_{k=1}^{n} W_{-2k+1} = \frac{-W_{-2n+2}+rW_{-2n+1}-uW_{-2n}+W_2-rW_1+uW_0}{r+t}.$$

Note that (c) is a special case of (b).

Proof.

(a) Using the recurrence relation

$$W_{-n+4} = rW_{-n+3}+sW_{-n+2}+tW_{-n+1}+uW_{-n}$$

i.e.

$$uW_{-n} = W_{-n+4}-rW_{-n+3}-sW_{-n+2}-tW_{-n+1}$$

we obtain

$$\begin{aligned}
uW_{-n} &= W_{-n+4} - rW_{-n+3} - sW_{-n+2} - tW_{-n+1} \\
uW_{-n+1} &= W_{-n+5} - rW_{-n+4} - sW_{-n+3} - tW_{-n+2} \\
uW_{-n+2} &= W_{-n+6} - rW_{-n+5} - sW_{-n+4} - tW_{-n+3} \\
&\vdots \\
uW_{-5} &= W_{-1} - rW_{-2} - sW_{-3} - tW_{-4} \\
uW_{-4} &= W_{0} - rW_{-1} - sW_{-2} - tW_{-3} \\
uW_{-3} &= W_{1} - rW_{0} - sW_{-1} - tW_{-2} \\
uW_{-2} &= W_{2} - rW_{1} - sW_{0} - tW_{-1} \\
uW_{-1} &= W_{3} - rW_{2} - sW_{1} - tW_{0}.
\end{aligned}$$

If we add the above equations by side by, we obtain

$$\begin{aligned}
u(\sum_{k=1}^{n} W_{-k}) &= (-W_{-n+3} - W_{-n+2} - W_{-n+1} - W_{-n} + W_3 + W_2 + W_1 + W_0 + \sum_{k=1}^{n} W_{-k}) \\
&\quad -r(-W_{-n+2} - W_{-n+1} - W_{-n} + W_2 + W_1 + W_0 + \sum_{k=1}^{n} W_{-k}) \\
&\quad -s(-W_{-n+1} - W_{-n} + W_1 + W_0 + \sum_{k=1}^{n} W_{-k}) - t(-W_{-n} + W_0 + \sum_{k=1}^{n} W_{-k}).
\end{aligned}$$

From the last equation we get (a).

(b) and (c) Using the recurrence relation

$$W_{-n+4} = rW_{-n+3} + sW_{-n+2} + tW_{-n+1} + uW_{-n}$$

i.e.

$$tW_{-n+1} = W_{-n+4} - rW_{-n+3} - sW_{-n+2} - uW_{-n}$$

we obtain

$$\begin{aligned}
tW_{-2n+1} &= W_{-2n+4} - rW_{-2n+3} - sW_{-2n+2} - uW_{-2n} \\
tW_{-2n+3} &= W_{-2n+6} - rW_{-2n+5} - sW_{-2n+4} - uW_{-2n+2} \\
tW_{-2n+5} &= W_{-2n+8} - rW_{-2n+7} - sW_{-2n+6} - uW_{-2n+4} \\
tW_{-2n+7} &= W_{-2n+10} - rW_{-2n+9} - sW_{-2n+8} - uW_{-2n+6} \\
&\vdots \\
tW_{-5} &= W_{-2} - rW_{-3} - sW_{-4} - uW_{-6} \\
tW_{-3} &= W_{0} - rW_{-1} - sW_{-2} - uW_{-4} \\
tW_{-1} &= W_{2} - rW_{1} - sW_{0} - uW_{-2}.
\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned}
t\sum_{k=1}^{n} W_{-2k+1} &= (-W_{-2n+2} - W_{-2n} + W_0 + W_2 + \sum_{k=1}^{n} W_{-2k}) - r(-W_{-2n+1} + W_1 + \sum_{k=1}^{n} W_{-2k+1}) \\
&\quad -s(-W_{-2n} + W_0 + \sum_{k=1}^{n} W_{-2k}) - u(\sum_{k=1}^{n} W_{-2k}).
\end{aligned} \tag{3.1}$$

Similarly, using the recurrence relation

$$W_{-n+4} = rW_{-n+3} + sW_{-n+2} + tW_{-n+1} + uW_{-n}$$

i.e.

$$tW_{-n} = W_{-n+3} - rW_{-n+2} - sW_{-n+1} - uW_{-n-1}$$

we obtain

$$\begin{aligned}
tW_{-2n} &= W_{-2n+3} - rW_{-2n+2} - sW_{-2n+1} - uW_{-2n-1} \\
tW_{-2n+2} &= W_{-2n+5} - rW_{-2n+4} - sW_{-2n+3} - uW_{-2n+1} \\
tW_{-2n+4} &= W_{-2n+7} - rW_{-2n+6} - sW_{-2n+5} - uW_{-2n+3} \\
tW_{-2n+6} &= W_{-2n+9} - rW_{-2n+8} - sW_{-2n+7} - uW_{-2n+5} \\
&\vdots \\
tW_{-8} &= W_{-5} - rW_{-6} - sW_{-7} - uW_{-9} \\
tW_{-6} &= W_{-3} - rW_{-4} - sW_{-5} - uW_{-7} \\
tW_{-4} &= W_{-1} - rW_{-2} - sW_{-3} - uW_{-5} \\
tW_{-2} &= W_{1} - rW_{0} - sW_{-1} - uW_{-3}.
\end{aligned}$$

If we add the above equations by side by, we get

$$\begin{aligned}
t\sum_{k=1}^{n} W_{-2k} &= (-W_{-2n+1} + W_1 + \sum_{k=1}^{n} W_{-2k+1}) - r(-W_{-2n} + W_0 + \sum_{k=1}^{n} W_{-2k}) \\
&\quad -s(\sum_{k=1}^{n} W_{-2k+1}) - u(W_{-2n-1} - W_{-1} + \sum_{k=1}^{n} W_{-2k+1}).
\end{aligned}$$

Since

$$W_{-1} = -\frac{t}{u}W_0 - \frac{s}{u}W_1 - \frac{r}{u}W_2 + \frac{1}{u}W_3$$

it follows that

$$\begin{aligned}
t\sum_{k=1}^{n} W_{-2k} &= (-W_{-2n+1} + W_1 + \sum_{k=1}^{n} W_{-2k+1}) - r(-W_{-2n} + W_0 + \sum_{k=1}^{n} W_{-2k}) \\
&\quad -s(\sum_{k=1}^{n} W_{-2k+1}) - u(W_{-2n-1} - (-\frac{t}{u}W_0 - \frac{s}{u}W_1 - \frac{r}{u}W_2 + \frac{1}{u}W_3) + \sum_{k=1}^{n} W_{-2k+1}).
\end{aligned} \tag{3.2}$$

Then, solving system (3.1)-(3.2) the required results of (b) and (c) follow.

Taking $r = s = t = u = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following proposition.

Proposition 3.2. If $r = s = t = u = 1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n} W_{-k} = \frac{1}{3}(-W_{-n+3} + W_{-n+1} + 2W_{-n} + W_3 - W_1 - 2W_0)$.

(b) $\sum_{k=1}^{n} W_{-2k} = \frac{1}{3}(W_{-2n+2} - 3W_{-2n+1} + 2W_{-2n} - 2W_{-2n-1} + 2W_3 - 3W_2 + W_1 - 4W_0)$.

(c) $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{3}(-2W_{-2n+2} + 3W_{-2n+1} - W_{-2n} + W_{-2n-1} - W_3 + 3W_2 - 2W_1 + 2W_0)$.

From the above proposition, we have the following corollary which gives linear sum formulas of Tetranacci numbers (take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2$).

Corollary 3.3. For $n \geq 1$, Tetranacci numbers have the following properties.

(a) $\sum_{k=1}^{n} M_{-k} = \frac{1}{3}(-M_{-n+3} + M_{-n+1} + 2M_{-n} + 1)$.

(b) $\sum_{k=1}^{n} M_{-2k} = \frac{1}{3}(M_{-2n+2} - 3M_{-2n+1} + 2M_{-2n} - 2M_{-2n-1} + 2)$.

(c) $\sum_{k=1}^{n} M_{-2k+1} = \frac{1}{3}(-2M_{-2n+2} + 3M_{-2n+1} - M_{-2n} + M_{-2n-1} - 1)$.

Taking $W_n = R_n$ with $R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7$ in the above proposition, we have the following corollary which presents linear sum formulas of Tetranacci-Lucas numbers.

Corollary 3.4. For $n \geq 1,$ Tetranacci-Lucas numbers have the following properties.

(a) $\sum_{k=1}^{n} R_{-k} = \frac{1}{3}(-R_{-n+3} + R_{-n+1} + 2R_{-n} - 2)$.

(b) $\sum_{k=1}^{n} R_{-2k} = \frac{1}{3}(R_{-2n+2} - 3R_{-2n+1} + 2R_{-2n} - 2R_{-2n-1} - 10)$.

(c) $\sum_{k=1}^{n} R_{-2k+1} = \frac{1}{3}(-2R_{-2n+2} + 3R_{-2n+1} - R_{-2n} + R_{-2n-1} + 8)$.

Taking $r = 2, s = t = u = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following proposition.

Proposition 3.5. If $r = 2, s = t = u = 1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n} W_{-k} = \frac{1}{4}(-W_{-n+3} + W_{-n+2} + 2W_{-n+1} + 3W_{-n} + W_3 - W_2 - 2W_1 - 3W_0)$.

(b) $\sum_{k=1}^{n} W_{-2k} = \frac{1}{8}(W_{-2n+2} - 5W_{-2n+1} + 6W_{-2n} - 3W_{-2n-1} + 3W_3 - 7W_2 + 2W_1 - 9W_0)$.

(c) $\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{8}(-3W_{-2n+2} + 7W_{-2n+1} - 2W_{-2n} + W_{-2n-1} - W_3 + 5W_2 - 6W_1 + 3W_0)$.

From the last proposition, we have the following corollary which gives linear sum formulas of fourth-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5$).

Corollary 3.6. For $n \geq 1,$ fourth-order Pell numbers have the following properties:

(a) $\sum_{k=1}^{n} P_{-k} = \frac{1}{4}(-P_{-n+3} + P_{-n+2} + 2P_{-n+1} + 3P_{-n} + 1)$.

(b) $\sum_{k=1}^{n} P_{-2k} = \frac{1}{8}(P_{-2n+2} - 5P_{-2n+1} + 6P_{-2n} - 3P_{-2n-1} + 3)$.

(c) $\sum_{k=1}^{n} P_{-2k+1} = \frac{1}{8}(-3P_{-2n+2} + 7P_{-2n+1} - 2P_{-2n} + P_{-2n-1} - 1)$.

Taking $W_n = Q_n$ with $Q_0 = 4, Q_1 = 2, Q_2 = 6, Q_3 = 17$ in the last proposition, we have the following corollary which presents linear sum formulas of fourth-order Pell-Lucas numbers.

Corollary 3.7. For $n \geq 1,$ fourth-order Pell-Lucas numbers have the following properties:

(a) $\sum_{k=1}^{n} Q_{-k} = \frac{1}{4}(-Q_{-n+3} + Q_{-n+2} + 2Q_{-n+1} + 3Q_{-n} - 5)$.

(b) $\sum_{k=1}^{n} Q_{-2k} = \frac{1}{8}(Q_{-2n+2} - 5Q_{-2n+1} + 6Q_{-2n} - 3Q_{-2n-1} - 23)$.

(c) $\sum_{k=1}^{n} Q_{-2k+1} = \frac{1}{8}(-3Q_{-2n+2} + 7Q_{-2n+1} - 2Q_{-2n} + Q_{-2n-1} + 13)$.

If $r = s = t = 1, u = 2$ then $(r + s + t + u - 1)(r - s + t - u + 1) = 0$ so we can't use Theorem 3.1 (b). In other words, the method of the proof Theorem 3.1 (b) can't be used to find $\sum_{k=1}^{n} W_{-2k}$ and $\sum_{k=1}^{n} W_{-2k+1}$.

Proposition 3.8. If $r = s = t = 1, u = 2$ then for $n \geq 1$ we have the following formula:

$$\sum_{k=1}^{n} W_{-k} = \frac{1}{4}(-W_{-n+3} + W_{-n+1} + 2W_{-n} + W_3 - W_1 - 2W_0).$$

Taking $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1$ in the last proposition, we have the following corollary which presents linear sum formula of fourth-order Jacobsthal numbers.

Corollary 3.9. For $n \geq 1,$ fourth order Jacobsthal numbers have the following property

$$\sum_{k=1}^{n} J_{-k} = \frac{1}{4}(-J_{-n+3} + J_{-n+1} + 2J_{-n}).$$

From the last proposition, we have the following corollary which gives linear sum formulas of fourth order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10$).

Corollary 3.10. For $n \geq 1,$ fourth order Jacobsthal-Lucas numbers have the following property

$$\sum_{k=0}^{n} j_{-k} = \frac{1}{4}(-j_{-n+3} + j_{-n+1} + 2j_{-n} + 5).$$

COMPETING INTERESTS

Author has declared that no competing interests exist.

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