Journal of Advances in Mathematics and Computer Science

33(5): 1-7, 2019; Article no.JAMCS.50977

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

On Nonsingularity of RSFPLR Circulant Matrices

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2019/v33i530191

Editor(s):

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Complete Peer review History: http://www.sdiarticle3.com/review-history/50977

Received: 08 June 2019

Accepted: 18 August 2019

Published: 26 August 2019

Original Research Article

Abstract

In this paper, we discuss the non-singularity of a row skew first-plus-last right (RSFPLR) circulant matrices with the first row $(a_1, a_2, \ldots, a_n)$, which is determined by entries of the first row. First, the sufficient condition for the matrix to be nonsingular is that, there exists an element a_{i_0} belonging to the first row, whose absolute value is greater than the sum of the corresponding power of 2 and the absolute values of the remaining $(n-1)$ elements, that is, $|a_{i_0}| > \sum_{i=1, i\neq i_0}^{n} 2^{i-i_0}|a_i|$. Moreover, we derive other sufficient conditions for judging the non-singularity of the matrix.

Keywords: RSFPLR circulant matrix; non-singularity; singularity.

2010 Mathematics Subject Classification: 15A09; 15B05.

1 Introduction

The circulant matrices have in recent years been extended in many directions. The $f(x)$-circulant matrices are natural extension of circulant matrices, and can be found in [1–12]. The $f(x)$-circulant

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matrix has a wide application, especially on the generalized cyclic codes [8]. The properties and structures of the $(x^n - x + 1)$- circulant matrices [9–12], which are called row skew first-plus-last right (RSFPLR) circulant matrices, are better than those of the general $f(x)$-circulant matrices, so there are good methods for discriminations its non-singularity.

Firstly, we introduce the RSFPLR circulant matrix in the following definition.

Definition 1.1. [10, 11] A matrix A=RSFPLRcircfr$(a_1, a_2, \ldots, a_n)$ of the form

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n \\ -a_n & a_1 + a_n & a_2 & \ddots & \ddots & a_{n-1} \\ -a_{n-1} & -a_n + a_{n-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & a_3 \\ -a_3 & -a_4 + a_3 & \ddots & \ddots & \ddots & a_2 \\ -a_2 & -a_3 + a_2 & -a_4 + a_3 & \ldots & -a_n + a_{n-1} & a_1 + a_n \end{pmatrix}_{n\times n} \tag{1.1}$$

is called a RSFPLR circulant matrix with the first row $(a_1, a_2, \ldots, a_n)$.

Note that the RSFPLR circulant matrix is a $(x^n - x + 1)$-circulant matrix [9–12].

Let $\Theta_{(-1,1)}$ be the basic RSFPLR circulant matrix, denoted by

$$\Theta_{(-1,1)} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & 1 \\ -1 & 1 & 0 & \ldots & 0 \end{pmatrix}_{n\times n}. \tag{1.2}$$

It is easily verified that $g(x) = x^n - x + 1$ has no repeated roots in its splitting field and $g(x) = x^n - x + 1$ is both the minimal polynomial and the characteristic polynomial of the matrix $\Theta_{(-1,1)}$.

In addition, a matrix A can be written in the form

$$A = f\left(\Theta_{(-1,1)}\right) = \sum_{i=1}^{n} a_i \Theta_{(-1,1)}^{i-1} \tag{1.3}$$

if and only if A is a RSFPLR circulant matrix, where the polynomial $f(x) = \sum_{i=1}^{n} a_i x^{i-1}$ is called the representer of the RSFPLR circulant matrix A. It is clear that A is a RSFPLR circulant matrix if and only if A commutes with the $\Theta_{(-1,1)}$, that is,

$$A\Theta_{(-1,1)} = \Theta_{(-1,1)}A. \tag{1.4}$$

Secondly, based on [1], we deduce the following lemma.

Lemma 1.1. *Let* $A =$ RSFPLRcircfr$(a_1, a_2, \ldots, a_n)$ *be a RSFPLR circulant matrix with the first row* $(a_1, a_2, \ldots, a_n)$. *Then* A *is singular if and only if there exists* $j_0 (1 \le j_0 \le n)$ *such that* $f(\omega_{j_0}) = 0$, *where* $f(x) = \sum_{i=1}^{n} a_i x^{i-1}$.

2 Main Results

Let $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ be a RSFPLR circulant matrix with the first row $(a_1, a_2, \ldots, a_n)$. We discuss the non-singularity on matrix A under different conditions in this section. At the same time, several corollaries are derived.

Theorem 2.1. *Let $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ be given as in (1.1). If there exists an $a_{i_0} \in \{a_1, a_2, \ldots, a_n\}$, such that*

$$|a_{i_0}| > \sum_{i=1, i \neq i_0}^{n} 2^{i-i_0} |a_i|, i = 1, ..., n, i \neq i_0, \tag{2.1}$$

then A is nonsingular.

Proof. If A is singular, then by Lemma 1.1, there exists $j_0 (1 \leq j_0 \leq n)$, such that

$$f(\omega_{j_0}) = \sum_{i=1}^{n} a_i (\omega_{j_0})^i = 0.$$

So

$$a_{i_0} (\omega_{j_0})^{i_0} = - \sum_{i=1, i \neq i_0}^{n} a_i (\omega_{j_0})^i.$$

Taking the absolute value of the above equation

$$|a_{i_0} (\omega_{j_0})^{i_0}| = |\sum_{i=1, i \neq i_0}^{n} a_i (\omega_{j_0})^i| \leq \sum_{i=1, i \neq i_0}^{n} |a_i| |\omega_{j_0}|^i,$$

we have

$$|a_{i_0}| \leq \sum_{i=1, i \neq i_0}^{n} |a_i| |\omega_{j_0}|^{i-i_0}.$$

Note that ω_{j_0} are the roots of the characteristic polynomial $g(x) = x^n - x + 1$ for matrix $\Theta_{(-1,1)}$, i.e. $(\omega_{j_0})^n - \omega_{j_0} + 1 = 0$. So we get from [13, Corollary 6.1.5] that

$$|\omega_{j_0}| \leq 2.$$

Hence

$$|a_{i_0}| \leq \sum_{i=1, i \neq i_0}^{n} 2^{i-i_0} |a_i|,$$

which contradicts to inequality (2.1). Therefore, A is nonsingular. □

Corollary 2.2. *Let $A = \mathrm{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ be given as in (1.1). If there exists an $a_{i_0} \in \{a_1, a_2, ..., a_n\}$, for any $i \neq i_0$,$1 \leq i \leq n$, such that*

$$|a_{i_0}| > (n-1)|a_i| \sqrt[n]{2^{i-i_0}}, \tag{2.2}$$

then A is nonsingular.

Proof. If A is singular, then by Lemma 1.1, there exists $j_0 (1 \leq j_0 \leq n)$, such that

$$f(\omega_{j_0}) = \sum_{i=1}^{n} a_i (\omega_{j_0})^i = 0.$$

So

$$a_{i_0}(\omega_{j_0})^{i_0} = -\sum_{i=1,i\neq i_0}^{n} a_i(\omega_{j_0})^i.$$

Taking the absolute value of the above equation

$$|a_{i_0}(\omega_{j_0})^{i_0}| = |\sum_{i=1,i\neq i_0}^{n} a_i(\omega_{j_0})^i| \leq \sum_{i=1,i\neq i_0}^{n} |a_i||\omega_{j_0}|^i,$$

we get

$$|a_{i_0}| \leq \sum_{i=1,i\neq i_0}^{n} |a_i||\omega_{j_0}|^{i-i_0}.$$

Note that ω_{j_0} are the roots of the characteristic polynomial $g(x) = x^n - x + 1$ for matrix $\Theta_{(-1,1)}$, i.e. $(\omega_{j_0})^n - \omega_{j_0} + 1 = 0$. So we get from [13, Corollary 6.1.5] that

$$|\omega_{j_0}| \leq 2.$$

Thus

$$|a_{i_0}| \leq \sum_{i=1,i\neq i_0}^{n} 2^{i-i_0}|a_i|.$$

Hence there exists k_0, such that

$$|a_{i_0}| \leq (n-1)|a_{k_0}| \sqrt[n]{2^{k_0-i_0}},$$

which contradicts to inequality (2.2). Therefore, A is nonsingular. □

Corollary 2.3. *Let* $A = \text{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ *be given as in (1.1). If there exists an* $a_{i_0} \in \{a_1, a_2, ..., a_n\}$, *for any* $i \neq i_0$, $1 \leq i \leq n$, *such that*

$$\frac{|a_i|}{|a_{i_0}|} < \frac{1}{(n-1)\sqrt[n]{2^{i-i_0}}},$$

then A *is nonsingular.*

Proof. The proof process similar to Corollary 2.2. □

Theorem 2.4. *Let* $A = \text{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ *be given as in (1.1). If*

$$|a_M| > \sum_{i=1,i\neq M}^{n} 2^{i-M}|a_i|, \tag{2.3}$$

then A *is nonsingular, where* $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Proof. The proof process similar to Theorem 2.1 □

Corollary 2.5. *Let* $A = \text{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ *be given as in (1.1). If for any* $i \neq M$,$1 \leq i \leq n$, *such that*

$$|a_M| > (n-1)2^{i-M}|a_i|, \tag{2.4}$$

then A *is nonsingular, where* $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Proof. If A is singular, then by Lemma 1.1, there exists $j_0(1 \leq j_0 \leq n)$, such that

$$f(\omega_{j_0}) = \sum_{i=1}^{n} a_i(\omega_{j_0})^i = 0.$$

So

$$a_M(\omega_{j_0})^M = -\sum_{i=1, i \neq M}^{n} a_i(\omega_{j_0})^i.$$

Taking the absolute value of the above equation

$$|a_M(\omega_{j_0})^M| = |\sum_{i=1 i \neq M}^{n} a_i(\omega_{j_0})^i| \leq \sum_{i=1, i \neq M}^{n} |a_i||\omega_{j_0}|^i,$$

we have

$$|a_M| \leq \sum_{i=1, i \neq M}^{n} |a_i||\omega_{j_0}|^{i-M}.$$

Note that ω_{j_0} are the roots of the characteristic polynomial $g(x) = x^n - x + 1$ for matrix $\Theta_{(-1,1)}$, i.e. $(\omega_{j_0})^n - \omega_{j_0} + 1 = 0$. So we get from [13, Corollary 6.1.5] that

$$|\omega_{j_0}| \leq 2.$$

Thus

$$|a_M| \leq \sum_{i=1, i \neq M}^{n} 2^{i-M}|a_i|$$

Hence there exists k_0, such that

$$|a_M| \leq (n-1)2^{k_0 - M}|a_{k_0}|,$$

which contradicts to inequality (2.4). Therefore A is nonsingular. □

Corollary 2.6. *Let* $A = \text{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ *be given as in (1.1). If for any* $i \neq M, 1 \leq i \leq n$*, such that*

$$\sum_{i=1 i \neq M}^{n} \frac{|a_i|}{|a_M|} 2^{i-M},$$

then A *is nonsingular, where* $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Proof. The proof process similar to Corollary 2.5. □

Corollary 2.7. *Let* $A = \text{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ *be given as in (1.1). If for any* $i \neq M, 1 \leq i \leq n$*, such that*

$$\frac{|a_i|}{|a_M|} < \frac{1}{(n-1)\sqrt[n]{2^{i-M}}},$$

then A *is nonsingular, where* $a_M = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

Proof. The proof process similar to Corollary 2.5. □

Theorem 2.8. *Let* $A = \text{RSFPLRcircfr}(a_1, a_2, \ldots, a_n)$ *be given as in (1.1). If there exists an* $a_{i_0} \in (a_1, a_2, ..., a_n)$*, such that*

$$|1 - a_{i_0}| < \frac{1}{n}, 2|a_i| < \frac{1}{n}, i = 1, ..., n, i \neq i_0,$$

then A *is nonsingular.*

Proof. Add the n inequalities of the both sides,

$$|1-a_{i_0}|+\sum_{i=1,i\neq i_0}^{n}2^{i-i_0}|a_i|<1.$$

Since

$$|1-a_{i_0}|\geq 1-|a_{i_0}|,$$

we have

$$|a_{i_0}|>\sum_{i=1,i\neq i_0}^{n}2^{i-i_0}|a_i|. \tag{2.5}$$

Therefore,the conclusion is clearly established based on Theorem 2.1 and (2.5). □

Corollary 2.9. *Let* $A=\mathrm{RSFPLRcircfr}(a_1,a_2,\ldots,a_n)$ *be given as in (1.1). If*

$$|1-a_M|<\frac{1}{n},\ 2|a_i|<\frac{1}{n},\ i=1,...,n,\ i\neq M,$$

then A *is nonsingular, where* $a_M=\max\{|a_1|,|a_2|,...,|a_n|\}$.

Proof. Add the n inequalities of the both sides,

$$|1-a_M|+\sum_{i=1,i\neq M}^{n}2^{i-M}|a_i|<1.$$

Since

$$|1-a_M|\geq 1-|a_M|,$$

we have

$$|a_M|>\sum_{i=1,i\neq M}^{n}2^{i-M}|a_i|.$$

According to Theorem 2.4, A is nonsingular. □

Theorem 2.10. *Let* $A=\mathrm{RSFPLRcircfr}(a_1,a_2,\ldots,a_n)$ *be given as in (1.1). If*

$$\sqrt{n[(1-a_M)^2+\sum_{i=1,i\neq M}^{n}|a_i|^2 2^{2(i-M)}]}<1, \tag{2.6}$$

then A *is nonsingular, where* $a_M=\max\{|a_1|,|a_2|,...,|a_n|\}$.

Proof. Since

$$\sqrt{\frac{(1-a_M)^2+\sum_{i=1,i\neq M}^{n}|a_i|^2 2^{2(i-M)}}{n}}\geq\frac{|1-a_M|+\sum_{i=1,i\neq M}^{n}|a_i|2^{i-M}}{n},$$

we have

$$\begin{aligned}\sqrt{n[(1-a_M)^2+\sum_{i=1,i\neq M}^{n}|a_i|^2 2^{2(i-M)}]}&\geq|1-a_M|+\sum_{i=1,i\neq M}^{n}|a_i|2^{i-M}\\&\geq 1-|a_M|+\sum_{i=1,i\neq M}^{n}|a_i|2^{i-M}\end{aligned}$$

By the inequality (2.6), we get

$$|a_M|\geq\sum_{i=1,i\neq M}^{n}|a_i|2^{i-M}.$$

According to Theorem 2.4, A is nonsingular. □

Acknowledgement

The research was Supported by the AMEP of Linyi University, China.

Competing Interest

The authors have declare that no competing interests exist.

References

[1] Jiang ZL, Xu ZB. Efficient algorithm for finding the inverse and group inverse of FLS r-circulant matrix. J. Appl. Math. Comput. 2005;18:45–57.

[2] Li J, Jiang ZL, Shen N. Explicit determinants of the Fibonacci RFPLR circulant and Lucas RFPLL circulant matrix. JP J. Algebra Number Theory Appl. 2013;28(2):167–179.

[3] Jiang ZL, Li J, Shen N. On the explicit determinants of the RFPLR and RFPLL circulant matrices involving Pell numbers in information theory. Commun. Comput. Inf. Sci. 2012;308:364-370.

[4] Tian ZP. Fast algorithm for solving the first plus last circulant linear system. J. Shandong Univ. Nat. Sci. 2011;46(12):96-103.

[5] Shen N, Jiang ZL, Li J. On explicit determinants of the RFMLR and RLMFL circulant matrices involving certain famous numbers. WSEAS Trans. Math. 2013;12(1):42-53.

[6] Jiang ZL, Shen N, Li J. On the explicit determinants of the RFMLR and RLMFL circulant matrices involving Jacobsthal numbers in communication. Proceedings of the 9th International Symposium on Linear Drives for Industry Applications. 2014;3:401-408.

[7] Tian ZP. Fast algorithms for solving the inverse problem of $AX = b$ in four different families of patterned matrices. Far East J. Appl. Math. 2011;52:1-12.

[8] David C. Regular representations of semisimple algebras, separable field extensions, group characters, generalized circulants, and generalized cyclic codes. Linear Algebra Appl. 1995;218:147-183.

[9] Jiang XY, Hong KC, Fu ZW. Skew cyclic displacements and decompositions of inverse matrix for an innovative structure matrix. J. Nonlinear Sci. Appl. 2017;10:4058-4070.

[10] Jiang XY, Hong KC. Explicit determinants of the k-Fibonacci and k-Lucas RSFPLR circulant matrix in codes, Inf. Comput. Appl. Springer, Berlin, Heidelberg, 2013:625-637.

[11] Jiang XY, Hong KC. Algorithms for finding inverse of two patterned matrices over $\mathbb{Z}_p$. Abstr. Appl. Anal. 2014;6. Article ID 840435

[12] Jiang XY, Hong KC. Exact determinants of some special circulant matrices involving four kinds of famous numbers. Abstr. Appl. Anal. 2014;12. Article ID 273680.

[13] Horn RA, Johnson CR. Matrix analysis. Cambridge University Press, New York; 1990.

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