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Distributional Chaoticity of the Minimal Subshift of Shift Operators

Yuanlin Chen[1], Tianxiu Lu[1,2*], Jiazheng Zhao[1]

[1]College of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, China
[2]Key Laboratory of Bridge Non-destruction Detecting and the Engineering Computing, Zigong, China
Email: cdcm03098915@163.com, *lubeeltx@163.com, 323070108114@stu.suse.edu.cn

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Abstract

This paper focus on the chaotic properties of minimal subshift of shift operators. It is proved that the minimal subshift of shift operators is uniformly distributional chaotic, distributional chaotic in a sequence, distributional chaotic of type k ($k \in \left\{1, 2, 2\frac{1}{2}, 3\right\}$), and $(0,1)$ -distribution.

Keywords

Shift Operators, Subshift, Distributional Chaoticity

1. Introduction

The shift operator is a distinct linear operator that displaces one or more bits forward (or backward) for each basis vector in the canonical orthogonal basis of Banach space or Frechet space. It has been widely used in many fields, for example, image processing ([1], 2015), chaotic encryption ([2], 2020), symbolic dynamical systems ([3], 2013), and dynamic physical systems ([4], 1993).

The shift operator is generally divided into two categories: unilateral shift operator and bilateral shift operator. The weighted shift operator is a generalization of the shift operator. Since Grosse-Erdmann ([5], 1999) closely linked the hypercyclicity of operators to the topological transitivity of dynamical systems, the study of chaotic properties of shift operators has attracted more and more attention, especially in symbolic spaces. In 2010, Queffelec ([6]) proved that the shift operator acting on the symbol space is topologically exact, so it is topologically mixing. And "topologically weakly mixing" is equivalent to "topologically transitive", "topologically transitive" is equivalent to "having a dense orbit". In 2013, Wu and Zhu ([7]) studied chaos generated by a class of weighted shift operators. Firstly, it is proved that the weighted shift operator is weakly mixing, transitive (or hypercyclic), and Devaney chaotic are equivalent to the separability of space. Moreover, this property is preserved under iteration. Then, it is obtained that the weighted shift operator is distributional chaotic and Li-Yorke sensitive. They also studied the dynamical properties of general weighted shift operators ([8]). It is proved that the weighted shift operator is uniformly distributional chaotic, and this property is maintained under iteration. In addition, it is proved that the principal measure of the weighted shift operator is equal to 1. Wang ([9], 2018) proved that there exists an uncountable invariant distributional irregular set of weighted shift operators on normed linear spaces $\Sigma(X)$, which generalizes the main results of [8].

The study of shift operators acting on symbolic dynamical systems is often more concerned with the chaotic properties of their subshifts. Constructing counterexamples is a common and highly important method in mathematical research. The unique representation of symbolic space subshifts provides a straightforward tool for constructing counterexamples in the study of dynamical systems. Consequently, symbolic space subshifts play a crucial role in exploring various chaotic processes. It is necessary to delve into the various intriguing mathematical properties of subshifts. Exploring the chaotic properties of minimal subshifts reveals the complex balance between order and unpredictability in dynamic systems. Consider the logistic map—a classic example of chaos. Through simple iterations of a mathematical function, it showcases chaotic behavior, exemplifying the butterfly effect: tiny initial changes lead to vastly different outcomes. These insights transcend conventional mathematics and physics, impacting fields like weather forecasting. By dissecting the chaotic properties of minimal subshifts, we gain effective tools to understand and harness interdisciplinary complexity. Jiang and Fa ([10], 1993) studied the subshifts of finite-type symbolic dynamical systems and proved that finite-type subshifts are chaotic in the sense of Li-Yorke. Inspired by [8] and [10], Liao and Fan construct a minimal subshift of the shift operator in [11]. It is proved that this minimal subshift is distributional chaotic, and its topological entropy is zero, so it is not topological chaotic. Furthermore, it is shown that positive topological entropy and distributional chaotic are not equivalent. Fu ([12], 2000) proved that the one-sided subshift generated by aperiodic recursive points is chaotic in the Robinson sense. In addition, if the subshift has a periodic point, then it has an infinite permutation set. Finally, some examples are given to discuss the topological entropy of these sub-displacements. Oprocha and Wilczynski ([13], 2007) proved the equivalence between distributional chaotic, chaotic in the sense of Li-Yorke, positive entropy and uncountable of subshifts. Some recent studies about subshifts see ([14] [15]) and others.

With the development of chaos, three types of distributional chaotic (DC1, DC2, DC3) are proposed by Balibrea in [16]. Subsequently, DC$\frac{1}{2}$, (p,q)-DC, uniformly distributional chaotic, distributional chaotic in a sequence $\{p_k\}$, and distributional chaotic in a sequence $\{p_k^*\}$ have been proposed. Therefore, a natural problem arises. Do the subshifts constructed above is uniformly distributional chaotic or the above type of distributional chaotic? This paper answers the above questions. To address disturbances or illusions in dynamic systems, the concept of measure centrality is proposed, dividing chaos into three different levels of complexity. The perspective of hierarchical chaos will contribute to a deeper understanding of chaotic systems. It indicates that all significant dynamical states of a system are manifested in its measure centrality, where the measure centrality of a minimal system is itself, hence discussing issues on such minimal systems is meaningful. In contrast to other studies on the chaotic properties of subshifts, this paper focuses on constructed subshifts, namely the study of the distribution chaotic properties of minimal subshifts. This work aims to comprehensively explore the distribution patterns of minimal subshifts, revealing their chaotic behavior and aiding in a further understanding of the chaotic properties of subshifts from a distributional chaos perspective. In Section 2, some basic concepts and definitions are introduced. In Section 3, some necessary lemmas are given first. Then, it is proved that the subshift $\sigma|M$ is uniformly distributional chaotic. So, $\sigma|M$ is DC1, DC2, DC $2\frac{1}{2}$, DC3 and $(0,1)$-DC. In addition, it is proved that $\sigma|M$ is distributional chaotic in a sequence $\{p_k\}$ and a sequence $\{p_k^*\}$.

2. Preliminaries

Let $(X,\|\cdot\|)$ be a Banach space on the real number field $\mathbb{R}$. And

$$\Sigma(X)=\{x=(x_0,x_1,\cdots)\,|\,x_i\in X,i\in\mathbb{N}\}.$$

Suppose that ~ be an equivalence relation on *X*. A family of sets consisting of all different ~ equivalence classes of *X* is called the quotient set of *X* with respect to ~, denoted by *X*/~.

A metric $d:\Sigma(X)\times\Sigma(X)\to\mathbb{R}$ is defined as

$$d(x,y)=\sum_{n=0}^{+\infty}\frac{1}{2^n}\frac{\|x_n-y_n\|}{1+\|x_n-y_n\|}$$

for any $x=(x_0,x_1,\cdots),\ y=(y_0,y_1,\cdots)$.

Obviously, d is a metric on $\Sigma(X)$ and $(\Sigma(X),d)$ is a compact metric space. The backward shift operator $\sigma:\Sigma(X)\to\Sigma(X)$ is defined by

$$\sigma(x_0,x_1,\cdots)=(x_1,x_2,\cdots)$$

for any $x=(x_0,x_1,\cdots)\in\Sigma(X)$. In other words, the backward shift operator is to move the symbol sequence x on the space $\Sigma(X)$ to the left one by one. If M is a closed set and $\sigma(M)\subset M$, then $\sigma|_M:M\to M$ is called a subshifts of σ.

A finite arrangement $A=x_0x_1\cdots x_{n-1}$ of symbols in X is called a symbol segment on *X*, and n is the length of *A*, denoted by $|A|=n$. If $B=y_0y_1\cdots y_{m-1}$ is another symbol segment, write

$$AB = x_0 x_1 \cdots x_{n-1} y_0 y_1 \cdots y_{m-1},$$

then AB is also a symbol segment. A is said to appear in B (or A appears in B), denoted as $A \prec B$, if $n \le m$ and there is $0 \le k \le m-n$, such that $y_{k+i} = x_i$, $0 \le i \le n-1$.

In fact, a mapping g from a symbol segment $x = x_0 x_1 \cdots$ to $\Sigma(X)$ can be established as $g(x) = (x_0, x_1, \cdots)$.

A subset

$$[A] = \{ y = (y_0, y_1, \cdots) \mid y_i = x_i, 0 \le i \le n-1 \}$$

is called a cylinder of $g(A)$.

For a given unit element $e \in X$, denote

$$\Sigma(D) = \{ x = (x_0, x_1, \cdots) \mid x_i \in D = \{0, e\}, i \in \mathbb{N} \}.$$

Let $\bar{A} = \bar{a}_0 \bar{a}_1 \cdots \bar{a}_{n-1}$ be the inverse of a symbol segment $A = a_0 a_1 \cdots a_{n-1}$ on D, where

$$\bar{a}_i = \begin{cases} 0, & \text{if } a_i = e, \\ e, & \text{if } a_i = 0, \end{cases} \quad i = 0, 1, \cdots, n-1.$$

Obviously, $|A| = |\bar{A}|$ and $\bar{\bar{A}} = A$. Now take a symbol segment $A_0 = 0$, let A_1 be an arrangement of A_0 and $\bar{A}_0$, that is, $A_1 = A_0 \bar{A}_0$ or $A_1 = \bar{A}_0 A_0$. Inductively, the symbol segments $A_1, A_2, \cdots$ can be defined. And for any $n \ge 1$, A_n exactly a finite arrangement of all members in the symbol segment set

$$\mathcal{K}_{n-1} = \{ K_0 K_1 \cdots K_{n-1} \mid K_i \in A_i, \bar{A}_i, 0 \le i \le n-1 \}.$$

For any $n \ge 0$, denote

$$\mathcal{L}_n = |A_0 A_1 \cdots A_n|.$$

Let $a = A_0 A_1 \cdots$, then $g(a) \in \Sigma(D)$. Denote $M = \omega(g(a), \sigma) \subset \Sigma(D)$, M is the ω-limit set of $g(a)$, obviously M is a closed set and $\sigma(M) \subset M$, *i.e.*, through the aforementioned construction, M is obtained, where σ acts as a shift operator on the closed set M. So $\sigma|_M : M \to M$ is a subshift of σ. In fact, Liao and Fan ([11], 1998) proved that it is also a minimal subshift of σ.

The concepts of several types of distributonal chaos are introduced below.

Let f be a continuous self-map on a metric space X. For any pair $(x, y) \in X \times X$ and for each $n \in \mathbb{N}$, the distribution function $F_{xy}^n : \mathbb{R} \to [0,1]$ is defined by

$$F_{xy}^n(t) = \frac{1}{n} \operatorname{card} \{ 1 \le i \le n : d(f^i(x), f^i(y)) < t \},$$

where card $\{A\}$ denotes the cardinality of the set A.

Put

$$F_{xy}^*(t) = \limsup_{n \to \infty} F_{xy}^n(t), \; F_{xy}(t) = \liminf_{n \to \infty} F_{xy}^n(t).$$

Then $F_{xy}^*(t)$ is called the upper distribution function, and $F_{xy}(t)$ the lower distribution function of x and y.

If the pair (x, y) satisfies

(DC1) $F_{xy}^{*}=1$ and $F_{xy}(\epsilon)=0$ for some $\epsilon>0$, or

(DC2) $F_{xy}^{*}=1$ and $F_{xy}(\epsilon)<1$ for any $\epsilon>0$ in an interval, or

(DC2$\frac{1}{2}$) there exist $c>0$ and $r>0$ such that $F_{xy}(t)<c<F_{xy}^{*}(t)$ for any $0<t<r$, or

(DC3) $F_{xy}(t)<F_{xy}^{*}(t)$ for any $t>0$ in an interval, or

((*p*, *q*)-DC) there exist $r>0$ and $0\le p\le q\le 1$ such that

$$F_{xy}(t)=p, F_{xy}^{*}(t)=q$$

for any $0<t<r$, then, (x,y) is called a distributional chaotic pair of type *k* ($k\in\left\{1,2,2\frac{1}{2},3\right\}$) or type (p,q) for *f*. The mapping *f* is said to be distributional chaotic of type *k* ($k\in\left\{1,2,2\frac{1}{2},3\right\}$) or type (p,q) if there exists an uncountable set $F\subset X$ such that every pair (x,y) of distinct points in *F* is a distributional chaotic pair of type *k* ($k\in\left\{1,2,2\frac{1}{2},3\right\}$) or type (p,q) for *f*. *f* is said to be uniformly distributional chaotic if there exist an uncountable set $F\subset X$ and an $\epsilon>0$ such that for every pair (x,y) of distinct points in *F*, $F_{xy}^{*}=1$ and $F_{xy}(\epsilon)=0$. It follows by the definition that, for a continuous map *f* of a compact metric space, DC1 implies DC2, and DC2 implies DC3.

Next, the definition of distributional chaotic in a sequence is given.

Definition 2.1 [17] Let *X* be a compact metric space, $\{m_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive integers, and $f:X\to X$ be a continuous map.

Then *f* is said to be distributional chaotic in a sequence $\{m_i\}_{i=1}^{\infty}$ if there is an uncountable subset $S\subset X$ such that for any two distinct points $x,y\in S$,

$$\limsup_{n\to\infty}\frac{1}{n}\operatorname{card}\left\{1\le i\le n: d\left(f^{m_i}(x),f^{m_i}(y)\right)<t\right\}=1$$

for any $t>0$ and

$$\liminf_{n\to\infty}\frac{1}{n}\operatorname{card}\left\{1\le i\le n: d\left(f^{m_i}(x),f^{m_i}(y)\right)<\epsilon\right\}=0$$

for some $\epsilon>0$.

And, a stronger chaotic description than distributional chaotic in a sequence is introduced, which will use the following concepts about density.

Definition 2.2 [17] Let $J=\{j_n\}_{n=1}^{\infty}$ be a strictly increasing positive integer sequence. If $\lim_{n\to\infty}\frac{n}{j_n}$ exists, then the limit is called the density of the sequence *J*, denoted by $\mu(J)$.

For a given positive integer sequence $J=\{j_n\}_{n=1}^{\infty}$, and a positive integer *m*, write

$$C_J(m)=\operatorname{card}\{n\ge 1\mid j_n\le m\}.$$

Obviously, if the density of the sequence is meaningful, then

$$\mu(J) = \lim_{m\to\infty} \frac{C_J(m)}{m}.$$

Definition 2.3 [17] Let $J = \{j_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers and $\{m_i\}_{i=1}^{\infty}$ is a subsequence of *J*. The upper limit

$$\limsup_{n\to\infty} \frac{\operatorname{card}\{i \ge 1 \mid m_i \le j_n\}}{n}$$

is the upper density of the sequence $\{m_i\}_{i=1}^{\infty}$ relative to the sequence *J*, denoted by $\mu^*\left(\{m_i\}_{i=1}^{\infty} / J\right)$.

Definition 2.4 [17] A sequence $J = \{j_n\}_{n=1}^{\infty}$ is said converge weakly to *j* if there exists a subsequence $\{m_i\}_{i=1}^{\infty}$ of natural number sequence with upper density 1 such that $\lim_{i\to\infty} j_{m_i} = j$, denoted by $\lim_{i\to\infty}^{*} j_i = j$.

Definition 2.5 [17] Let *X* be a compact metric space, $\{m_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive integers, and $f: X \to X$ be a continuous map. *f* is said to be distributional chaotic in a sequence $\{m_i^*\}_{i=1}^{\infty}$ if there exists an uncountable set $S \subset X$ such that for any positive integer *l*, there exist *l* distinct points $b_1, b_2, \cdots, b_l \in X$ such that for any finite subset $A = \{a_1, a_2, \cdots, a_l\}$ of *l* points in *S*, and for any mapping $F: A \to \{b_1, b_2, \cdots, b_l\}$, the

$$\lim_{i\to\infty}^{*} f^{m_i}(a) = F(a)$$

for any $a \in A$.

From the definition of several distributional chaos mentioned above, uniformly distributional chaotic is stronger than DC1, because the number ϵ that appears in its definition does not depend on the pair (x, y). And DC1 implies DC2, DC2 implies DC $2\frac{1}{2}$, DC $2\frac{1}{2}$ implies DC3, *i.e.*,

$$\mathrm{DC1} \subset \mathrm{DC2} \subset \mathrm{DC2}\frac{1}{2} \subset \mathrm{DC3},$$

which in turn does not hold. Distributional chaos is obviously distributed in accordance with the natural number sequence, otherwise it does not necessarily hold.

3. Main Result

Lemma 3.1 [11] For any $n > 0$, $\mathcal{L}_n = (2^n + 1)\mathcal{L}_{n-1}$.

Proof. According to the definition, it can be directly verified.

Lemma 3.2 [11] For any $n > 0$, $a = A_0 A_1 \cdots$ is an infinite arrangement of symbol segments in $\mathcal{K}_n$.

Proof. This lemma can be found in [11]. For the completeness of the paper, we provide its proof. The following primarily consists of proving Lemma 3.2 through mathematical induction.

For any given $n > 0$, by definition, A_{n+1} and $\overline{A}_{n+1}$ are finite arrangement of symbol segments in $\mathcal{K}_n$. Suppose for some $k \ge 0$, it has been proved that A_{n+1}, $\overline{A}_{n+1}$, A_{n+2}, $\overline{A}_{n+2}$, $\cdots$, A_{n+k} and $\overline{A}_{n+k}$ are all the finite symbol segments in $\mathcal{K}_n$ has been proved. Since A_{n+k+1} and $\overline{A}_{n+k+1}$ are both the finite symbol segments of the form $K_0K_1\cdots K_n\cdots K_{n+k}$ in $\mathcal{K}_n$, where

$$K_0K_1\cdots K_n\in\mathcal{K}_n,\ K_{n+i}\in\left\{A_{n+i},\overline{A}_{n+i}\right\},1\le i\le k,$$

they are also the finite symbol segments in $\mathcal{K}_n$. In this way, it is proved that for each $m>n$, A_m and $\overline{A}_m$ are finite symbol segments in $\mathcal{K}_n$. By the definition of *a*, one can get that Lemma 3.2 holds.

Lemma 3.3 There is an uncountable subset $S\subset\Sigma(D)$ such that for any two distinct points $x=(x_0,x_1,\cdots),y=(y_0,y_1,\cdots)\in S$, $x_k=y_k$ for infinitely many *k* and $x_l\ne y_l$ for infinitely many *l*.

Proof. The key to proving Lemma 3.3 lies in defining an equivalence relation on the set $\Sigma(D)$, thereby obtaining uncountably many equivalence classes. Selecting one element from each equivalence class results in the construction of an uncountable set *S* satisfying the given conditions.

For any $x=(x_0,x_1,\cdots),y=(y_0,y_1,\cdots)\in\Sigma(D)$, define a relation on the set $\Sigma(D)$, denoted by $x\sim y$, if only for a finite number of *k*, $x_k=y_k$, or only for a finite number of *l*, $x_l\ne y_l$. It can be verified that ~ is an equivalence relation on the set $\Sigma(D)$. Let $x\in\Sigma(D)$, according to the above equivalence relation, it is not difficult to get a countable set $\left\{y\in\Sigma(D)\mid y\sim x\right\}$, so the quotient set $\Sigma(D)/\sim$ is uncountable.

Therefore, one can take a representative element in each equivalence class of an uncountable set $\Sigma(D)/\sim$ to form a subset of $\Sigma(D)$, denoted by *S*. Then, *S* is an uncountable set satisfying the condition.

Lemma 3.4 [18] Let *X* be a compact metric space and $f:X\to X$ be a continuous map. If there are two nonempty descending closed set sequences $\{U_i\}$, $\{V_i\}$ in *X* and a positive integer sequence $\{m_i\}_{i=1}^{\infty}$ such that

1) $\bigcap_{i=1}^{\infty}U_i=\{a\},\bigcap_{i=1}^{\infty}V_i=\{b\}$ and $a\ne b$;

2) $f^{m_i}(U_i)\cap f^{m_i}(V_i)\supset U_{i+1}\cup V_{i+1}$ for any $i=1,2,\cdots$,

then *f* is distributional chaotic in the sequence $\{p_k\}_{k=1}^{\infty}$, where

$$p_k=\sum_{i=1}^{k}m_i\left(k\in\mathbb{N}\right).$$

Proof. This conclusion can be found in [18]. For the completeness of the paper, we provide its proof. The proof is given by constructing a sequence of positive integers $\{r_i\}_{i=1}^{\infty}$ and defining a mapping φ from a set *E* to $\mathcal{W}$.

Let $\mathcal{W}=\left\{\{W_i\}\mid W_i\in\{U_i,V_i\},i=1,2,\cdots\right\}$, then for any $\{W_i\}\in\mathcal{W}$, there exists $x\in W_1$, such that for any $k\ge1$, $f^{p_k}(x)\in W_{k+1}$ holds, where $p_k=\sum_{i=1}^{k}m_i\ (k\in\mathbb{N})$.

The positive integer sequence $\{r_i\}_{i=1}^{\infty}$ is selected, where $r_1=1$, $r_{n+1}=(2^n+1)r_n$, $n\ge1$. Let $S\subset\Sigma(D)$ be uncountable, such that for any different points two distinct points $x=(x_0,x_1,\cdots),y=(y_0,y_1,\cdots)\in S$, there are infinitely many *m* such that $x_m\ne y_m$, and there are infinitely many *n* such that $x_n\ne y_n$, by Lemma 3.3, such a set *E* exists. Define $\varphi:E\to\mathcal{W}$ by $\varphi(x)=\{W_j\}$ for each $x=(x_0,x_1,\cdots)\in S$, where

$$W_1 = \begin{cases} U_1, & \text{if } x_0 = 0, \\ V_1, & \text{if } x_0 = e, \end{cases}$$

and for any $n > 1$, if $r_{n-1} < j \le r_n$, the

$$W_j = \begin{cases} U_j, & \text{if } x_j = 0, \\ V_j, & \text{if } x_j = e. \end{cases}$$

For each $W = \{W_j\} \in \varphi(E)$, it can be seen that there must be $x_w \in W_1 \subset X$ such that when $k \ge 1$, $f^{p_k}(x_w) \in W_{k+1}$ holds. Let $F = \{x_w \mid W \in \varphi(E)\}$. Since φ is injective, E is uncountable, so $\varphi(E)$ is uncountable, and then F is uncountable. Therefore, it is only necessary to verify that the points in F satisfy the two conditions of Definition 2.1. Let $x, y \in F$, $x \ne y$, there exists $\{W_j\}, \{Z_j\}$, such that when $k \ge 1$, there are $f^{p_k}(x) \in W_{k+1}$ and $f^{p_k}(y) \in Z_{k+1}$.

On the one hand, there exists $n_i \to +\infty$, such that for each *i*, when $r_{n_i-1} < j \le r_{n_i}$, W_j and Z_j are always different. Take $\varepsilon = \frac{1}{2} d(a,b)$, when *i* sufficiently large, and $r_{n_i-1} < j \le r_{n_i}$, $d(U_j, V_j) > \varepsilon$ holds. In particular, $d\left(f^{p_j}(x), f^{p_j}(y)\right) > \varepsilon$ holds. Then, for sufficiently large *i*, there is

$$\frac{1}{r_{n_i}} \sum_{k=1}^{r_{n_i}} \chi_{[0,\varepsilon)}\left(d\left(f^{p_k}(x), f^{p_k}(y)\right)\right) \le \frac{r_{n_i-1}}{r_{n_i}} = \frac{r_{n_i-1}}{\left(2^{n_i-1}+1\right) r_{n_i-1}} \to 0 \quad (i \to \infty).$$

This indicates that $F_{xy}(\varepsilon) = 0$.

On the other hand, $x \ne y$ also implies that there exists $n_j \to +\infty$ such that for each *j*, when $r_{n_j-1} < j \le r_{n_j}$, there is always $W_j = Z_j$, that is, W_j and Z_j are either U_j or V_j at the same time. For any $t > 0$, there exists a sufficiently large *j* such that when $r_{n_j-1} < j \le r_{n_j}$, $d(U_j, V_j) < t$ holds. In particular, $d\left(f^{p_j}(x), f^{p_j}(y)\right) < t$ holds. Then, for sufficiently large $j \; (j \to \infty)$, one can get

$$\frac{1}{r_{n_j}} \sum_{k=1}^{r_{n_j}} \chi_{[0,t)}\left(d\left(f^{p_k}(x), f^{p_k}(y)\right)\right) \ge \frac{r_{n_j} - r_{n_j-1}}{r_{n_j}} = 1 - \frac{r_{n_j-1}}{r_{n_j}} = 1 - \frac{r_{n_j-1}}{\left(2^{n_j-1}+1\right) r_{n_j-1}} \to 1.$$

Thus $F_{xy}^*(t) = 1$. In summary, the points in the uncountable set F satisfy the two conditions in Definition 2.1. Therefore, f is distributional chaotic in the sequence $\{p_k\}_{k=1}^{\infty}$, where $p_k = \sum_{i=1}^{k} m_i \; (k \in \mathbb{N})$.

Lemma 3.5 [17] Let X be a compact metric space and $f: X \to X$ be a continuous map. If there are l closed set sequences $\{B_i^j\}_{i=1}^{\infty} \; (j = 1, 2, \cdots, l)$ in X and a positive integer sequence $\{m_i\}_{i=1}^{\infty}$, such that

1) $\bigcap_{i=1}^{\infty} B_i^j = \{b^j\}$, $j = 1, 2, \cdots, l$ and $b^{j_1} \ne b^{j_2}$ for $j_1 \ne j_2$,

2) $\bigcap_{j=1}^{l} f^{m_i}(B_i^j) \supset \bigcup_{j=1}^{k} B_{i+1}^j$, for any $i = 1, 2, \cdots$.

Let $p_k = \sum_{i=1}^{k} m_i \; (k \in \mathbb{N})$, then f is distributional chaotic in the sequence $\{p_k^*\}_{k=1}^{\infty}$.

Proof. This result can be found in [17]. For the completeness of the paper, we provide its proof. Constructing an appropriate sequence of positive integers and defining a key mapping are the difficulties and key points of the whole proof.

Let $\mathcal{C} = \left\{\{C_i\} \mid C_i \in \left\{B_i^1, B_i^2, \cdots, B_i^l\right\}, i = 1, 2, \cdots\right\}$, then for any $\{C_i\} \in \mathcal{C}$, there exists $x \in C_1$, such that for any $k \ge 1$, $f^{p_k}(x) \in C_{k+1}$ holds, where $p_k = \sum_{i=1}^{k} m_i$ $(k \in \mathbb{N})$. The positive integer sequence $\{r_i\} = \{n!\}$ is selected, for any positive integer *s*, there must be an uncountable subset E_s (by [17]) of the symbol space E with s symbols $1, 2, \cdots, s$ such that for any s distinct points $w^1, w^2, \cdots, w^s$ in E_s and any mapping

$$\phi : \left\{w^1, w^2, \cdots, w^s\right\} \to \{1, 2, \cdots, s\}.$$

Obviously, there are s^s such mappings, denoted by $\phi^l\left(l = 1, 2, \cdots, s^s\right)$. Then there exists a sequence $\left\{N_i^l\right\}$ such that

$$w^j\left(N_1^l\right) = w^j\left(N_2^l\right) = \cdots = \phi^l\left(w^j\right).$$

Define $\varphi : E_s \to \mathcal{C}$ by $\varphi(w) = \left\{C_j\right\}$ for each $w = w_1 w_2 \cdots \in E_s$, where $C_1 = B_1^{w_1}$ and for any $n > 1$, if $r_{n-1} < s \le r_n$, the $C_s = B_s^{w_n}$. For each $C = \left\{C_j\right\} \in \varphi(E_s)$, it can be seen that there must be $x_c \in C_1 \subset X$ such that when $s \ge 1$, $f^{p_s}(x_c) \in C_{s+1}$ holds. Let $D = \left\{x_c \mid C \in \varphi(E_s)\right\}$. Since φ is injective, E_s is uncountable, so $\varphi(E_s)$ is uncountable, and then D is uncountable.

Now we assert that D is distributional scrambled set in the sequence $\left\{p_k^*\right\}_{k=1}^{\infty}$.

Indeed, for any subset $D_1 = \left\{x_1, x_2, \cdots, x_s\right\}$ with *s* different points in *D*, there exists $\left\{C_s^j\right\}$ such that for any $s \ge 1$, $f^{p_s}\left(x_j\right) \in C_{s+1}^j\left(j = 1, 2, \cdots, s\right)$ holds. For any $F : D_1 \to \left\{b^1, b^2, \cdots, b^k\right\}$, take ϕ^{l_0}, such that $b^{\phi^{l_0}\left(w^j\right)} = F\left(x_j\right)$ $\left(j = 1, 2, \cdots, s\right)$. Then, when $r_{N_{i-1}^{l_0}} < s \le r_{N_i^{l_0}}$, there are $f^{p_s}\left(x_j\right) \in B_s^{\phi^{l_0}\left(w^j\right)}$.

Thus, $\lim_{s\to\infty}^{*} f^{p_s}\left(x_j\right) = \left(x_j\right)$ for any $j = 1, 2, \cdots, s$. By Definition 2.5, *f* is distributional chaotic in the sequence $\left\{p_k^*\right\}_{k=1}^{\infty}$.

The following are the main conclusions of this study.

Theorem 3.1 The subshift operator $\sigma|_M$ is uniformly distributional chaotic.

Proof. The key uncountable set *S* can be obtained by lemma 3.3, and a mapping *f* satisfying certain conditions is defined on S, and then the set *F* is constructed according to the following construction method. It can be proved that *F* is a distributional ϵ-scrambled set of $\sigma|_M$ with $\epsilon = \frac{1}{2}$.

Let *S* be an uncountable subset of $\Sigma(D)$ such that for any two distinct points $x = (x_0, x_1, \cdots), y = (y_0, y_1, \cdots) \in S$, $x_k = y_k$ for infinitely many k and $x_l \ne y_l$ for infinitely many *l*.

Denote a mapping *f*, such that $f(x) = K_0 K_1 \cdots$ for any $x = (x_0, x_1, \cdots) \in S$, where

$$K_i = \begin{cases} A_i, & \text{if } x_i = e, \\ \overline{A_i}, & \text{if } x_i = 0, \end{cases} \quad i = 0, 1, \cdots$$

Let $h = g \circ f$, $F = h(S)$ and $\mathcal{L}_n = |K_0 K_1 \cdots K_n|$. For a fixed $i \in \mathbb{N}$ and arbitrarily selected $K_j\,(0 \le j \le i)$, there always have $b_i = K_0 K_1 \cdots K_i \prec A_{i+1} \prec a$. Therefore, there exists $k \ge 0$, such that the first $\mathcal{L}_i$ symbols of $\sigma^k(a)$ are b_i. That is to say, for a fixed $i \in \mathbb{N}$, there exists a $k \ge 0$ such that the front $\mathcal{L}_i$ components of $\sigma^k(g(a))$ are $g(b_i)$. This shows that for any $x \in E$, $h(x) \in \omega(g(a), \sigma) = M$. Therefore, $F \subset M$, and because h is injective, S is an uncountable set, so S is also an uncountable set.

Assume that F is a distributional ϵ-scrambled set of $\sigma|_M$ with $\epsilon = \frac{1}{2}$. For any pair $x, y \in F$ with $x \ne y$, let $\bar{x} = g^{-1}(x)$ and $\bar{y} = g^{-1}(y)$. Without loss of generality, we assume $\bar{x} = B_0 B_1 \cdots$ and $\bar{y} = C_0 C_1 \cdots$, where $B_i, C_i \in \{A_i, \bar{A}_i\}$, $i = 0,1,\cdots$. According to the construction of S, there exist a subsequence $\{B_{k_i}\}_{i=1}^{\infty} \subset \{B_0, B_1, B_2, \cdots\}$ such that $B_{k_i} = C_{k_i}\,(i \in \mathbb{N})$ and a subsequence $\{B_{l_i}\}_{i=1}^{\infty} \subset \{B_0, B_1, B_2, \cdots\}$ such that $B_{l_i} = \bar{C}_{l_i}\,(i \in \mathbb{N})$.

First, it is easy to see that the first $\mathcal{L}_{k_i} - m$ components of $\sigma^m(x)$ and $\sigma^m(y)$ coincide correspondingly for $\mathcal{L}_{k_i - 1} \le m \le \frac{\mathcal{L}_{k_i}}{2}$. So

$$d(\sigma^m(x), \sigma^m(y)) = \sum_{n=m}^{+\infty} \frac{1}{2^n} \frac{\|x_n - y_n\|}{1 + \|x_n - y_n\|} = \sum_{n=\mathcal{L}_{k_i} - m}^{+\infty} \frac{1}{2^n} \frac{\|x_n - y_n\|}{1 + \|x_n - y_n\|}$$
$$\le \sum_{n=\mathcal{L}_{k_i} - m}^{+\infty} \frac{1}{2^n} \le \sum_{n=\frac{\mathcal{L}_{k_i}}{2}}^{+\infty} \frac{1}{2^n} = \frac{1}{2^{\frac{\mathcal{L}_{k_i}}{2}+1}}.$$

Thus, for a given $t > 0$, there exists a positive integer N such that $d(\sigma^m(x), \sigma^m(y)) < t$ for any $m > N$ and any $\mathcal{L}_{k_i - 1} \le m \le \frac{\mathcal{L}_{k_i}}{2}$. Then,

$$F_{xy}^*(t) = \limsup_{n \to \infty} \frac{1}{n} \operatorname{card}\left\{1 \le m \le n : d(\sigma^m(x), \sigma^m(y)) < t\right\}$$
$$\ge \limsup_{i \to \infty} \frac{1}{\frac{\mathcal{L}_{k_i}}{2}} \operatorname{card}\left\{1 \le m \le \frac{\mathcal{L}_{k_i}}{2} : d(\sigma^m(x), \sigma^m(y)) < t\right\}$$
$$\ge \limsup_{i \to \infty} \frac{\frac{\mathcal{L}_{k_i}}{2} - \mathcal{L}_{k_i - 1}}{\frac{\mathcal{L}_{k_i}}{2}} = \limsup_{i \to \infty} \left(1 - \frac{2}{2^{k_i} + 1}\right) = 1.$$

Second, it can be obtained that $x_m \ne y_m \in D = \{0, e\}$ for $\mathcal{L}_{l_i - 1} \le m \le \mathcal{L}_{l_i}$. Then

$$d(\sigma^m(x), \sigma^m(y)) \ge \frac{\|x_m - y_m\|}{1 + \|x_m - y_m\|} = \frac{1}{2}.$$

So,

$$F_{xy}\left(\frac{1}{2}\right) = \liminf_{n \to \infty} \frac{1}{n} \operatorname{card}\left\{1 \le m \le n : d(\sigma^m(x), \sigma^m(y)) < \frac{1}{2}\right\}$$
$$\le \liminf_{i \to \infty} \frac{1}{\mathcal{L}_{l_i}} \operatorname{card}\left\{1 \le m \le \mathcal{L}_{l_i} : d(\sigma^m(x), \sigma^m(y)) < \frac{1}{2}\right\}$$
$$\le \liminf_{i \to \infty} \frac{\mathcal{L}_{l_i - 1}}{\mathcal{L}_{l_i}} = \liminf_{i \to \infty} \frac{1}{2^{l_i} + 1} = 0.$$

Hence, $\sigma|_M$ is uniformly distributional chaotic.

This proof has been completed.

Theorem 3.2 $\sigma|_M$ is distributional chaotic of type k ($k \in \left\{1, 2, 2\frac{1}{2}, 3\right\}$) and is (p,q)-distribution (where $p=0, q=1$).

Proof. This theorem can be obtained by Theorem 3.1. Since $\sigma|_M$ is uniformly distributional chaotic, then $\sigma|_M$ is DC1. So, $\sigma|_M$ is also DC2 and DC3.

According to the Theorem 3.1, there exist an uncountable set $F \subset M$ and a $\epsilon = \frac{1}{2} > 0$ such that, for every pair (x, y) of distinct points in *F*, $F_{xy}^* = 1$ and $F_{xy}\left(\frac{1}{2}\right) = 0$. Take $r = \frac{1}{2}$, for any $0 < t < r$,

$$\begin{aligned}
F_{xy}(t) &= \liminf_{n\to\infty} \frac{1}{n} \operatorname{card}\left\{1 \le m \le n : d\left(\sigma^m(x), \sigma^m(y)\right) < t\right\} \\
&\le \liminf_{n\to\infty} \frac{1}{n} \operatorname{card}\left\{1 \le m \le n : d\left(\sigma^m(x), \sigma^m(y)\right) < \frac{1}{2}\right\} \\
&\le \liminf_{i\to\infty} \frac{1}{\mathcal{L}_{l_i}} \operatorname{card}\left\{1 \le m \le \mathcal{L}_{l_i} : d\left(\sigma^m(x), \sigma^m(y)\right) < \frac{1}{2}\right\} \\
&\le \liminf_{i\to\infty} \frac{\mathcal{L}_{l_i - 1}}{\mathcal{L}_{l_i}} = \liminf_{i\to\infty} \frac{1}{2^{l_i} + 1} = 0.
\end{aligned}$$

Then, for every pair (x, y) of distinct points in *F*, there exist $c > 0$ and $r = \frac{1}{2} > 0$ such that $F_{xy}(t) = 0 < c < F_{xy}^*(t) = 1$ for any $0 < t < r$. Thus, $\sigma|_M$ is DC $2\frac{1}{2}$. Take $p = 0$ and $q = 1$, for every pair (x, y) of distinct points in *F*, $F_{xy}(t) = 0 = p$ and $F_{xy}^*(t) = 1 = q$ for any $0 < t < r$. Thus, $\sigma|_M$ is $(0,1)$-DC.

This proof has been completed.

Theorem 3.3 The subshift operator $\sigma|_M$ is distributional chaotic in a sequence $\{p_k\}_{k=1}^{\infty}$.

Proof. It is proved that only the sequence satisfying the condition can be constructed. For a given $x = (x_0, x_1, \cdots) \in \Sigma(D)$, let $k = K_0 K_1 \cdots$, $\bar{k} = \bar{K}_0 \bar{K}_1 \cdots$, and $\mathcal{L}_n = |K_0 K_1 \cdots K_n|$, where

$$K_i = \begin{cases} A_i, & \text{if } x_i = e, \\ \overline{A_i}, & \text{if } x_i = 0, \end{cases} \quad i = 0, 1, \cdots$$

Obviously, $k \ne \bar{k}$. Denote

$$U_n = [K_0 K_1 \cdots K_{n-1}] \cap M \quad \text{and} \quad V_n = [\bar{K}_0 \bar{K}_1 \cdots \bar{K}_{n-1}] \cap M,$$

where $n = 1, 2, \cdots$, then it is easy to see that

$$\bigcap_{n=1}^{\infty} U_n = K_0 K_1 \cdots = k, \quad \bigcap_{n=1}^{\infty} V_n = \bar{K}_0 \bar{K}_1 \cdots = \bar{k},$$

and

$$\sigma^{\mathcal{L}_{n-1}}(U_n) \cap \sigma^{\mathcal{L}_{n-1}}(V_n) = M \supset U_{n+1} \cup V_{n+1}, n = 1, 2, \cdots.$$

Let $p_k = \sum_{n=1}^{k} \mathcal{L}_{n-1} \left(k \in \mathbb{N}\right)$, according to Lemma 3.4, $\sigma|_M$ is distributional chaotic in the sequence $\{p_k\}_{k=1}^{\infty}$.

This proof has been completed.

Theorem 3.4 The subshift operator $\sigma|_M$ is distributional chaotic in a sequence $\left\{p_k^*\right\}\left(k = 1, 2, \cdots\right)$.

Proof. Similar to the proof of Theorem 3.3, it is sufficient to construct a sequence that satisfies the condition.

For a given $x = \left(x_0, x_1, \cdots\right) \in \Sigma\left(D\right)$, let $b = K_0 K_1 K_2 \cdots$ and

$$b_1 = \bar{K}_0 K_1 K_2 \cdots, b_2 = K_0 \bar{K}_1 K_2 \cdots, \cdots, b_l = K_0 K_1 \cdots K_{l-2} \bar{K}_{l-1} K_l \cdots,$$

$$\mathcal{L}_j = \left|K_0 K_1 \cdots K_j\right|,$$

where

$$K_i = \begin{cases} A_i, & \text{if } x_i = e, \\ \overline{A_i}, & \text{if } x_i = 0, \end{cases} \quad i = 0, 1, \cdots$$

Obviously, $b_1 \neq b_2 \neq \cdots \neq b_l$. Denote

$$B_1^j = \left[\bar{K}_0 K_1 K_2 \cdots K_{j-1}\right] \cap M, B_2^j = \left[K_0 \bar{K}_1 K_2 \cdots K_{j-1}\right] \cap M, \cdots,$$
$$B_l^j = \left[K_0 K_1 \cdots K_{l-2} \bar{K}_{l-1} K_l \cdots K_{j-1}\right] \cap M,$$

where $j = 1, 2, \cdots$, then it is easy to see that

$$\bigcap_{i=1}^{\infty} B_i^j = \left\{b^j\right\}\left(i = 1, 2, \cdots, l\right), \quad b^{j_1} \neq b^{j_2} \quad \text{for} \quad j_1 \neq j_2$$

and

$$\bigcap_{j=1}^{l} \sigma^{\mathcal{L}_{j-1}}\left(B_i^{j-1}\right) = M \supset \bigcup_{j=1}^{k} B_{i+1}^j$$

for any $i = 1, 2, \cdots$.

Let $p_k = \sum_{j=1}^{k} \mathcal{L}_{j-1} \left(k \in \mathbb{N}\right)$, according to Lemma 3.5, $\sigma|_M$ is distributional chaotic in a sequence $\left\{p_k^*\right\}\left(k = 1, 2, \cdots\right)$.

This proof has been completed.

4. Conclusion

This study investigated distributional chaoticity of minimal subshift of shift operators. The conclusions involved include DC1, DC2, DC3, DC $2\frac{1}{2}$, (p, q)-distributional chaos, uniformly distributional chaos, and distributional chaotic in a sequence. Compared with other literatures, the contents in this paper are more comprehensive.

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Authors and Affiliations

The authors Yuanlin Chen and Jiazheng Zhao is in Sichuan University of Science and Engineering. The corresponding author Tianxiu Lu has two affiliations, that is, Sichuan University of Science and Engineering and Key Laboratory of Bridge Non-destruction Detecting and the Engineering Computing.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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