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Blow-up Rates of Large Solutions for Quasilinear Elliptic Equations

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Abstract

In this paper, we mainly study the boundary behavior of solutions to boundary blow-up quasilinear elliptic problem

$$\begin{cases} \text{div}(|\nabla u|^{m-2}\nabla u) = b(x)f(u), & x \in \Omega, \\ u|_{\partial\Omega} = +\infty, \end{cases}$$

where Ω is a bounded domain with smooth boundary in R^N, $m > 1$, $b \in C^\alpha(\bar{\Omega})$ which is positive in Ω and maybe vanishing on the boundary and rapidly varying near the boundary.

Keywords: Large solutions, quasilinear elliptic equation, boundary blow-up, asymptotic behavior. solution.

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1 Introduction and the main results

In this paper, we plan to investigate the exact asymptotic behavior of solutions near the boundary for the following problem

$$\begin{cases} \text{div}(|\nabla u|^{m-2}\nabla u) = b(x)f(u), & x \in \Omega, \\ u|_{\partial\Omega} = +\infty, \end{cases} \tag{1.1}$$

where the last condition means that $u(x) \to +\infty$ as $d(x) = \text{dist}(x, \partial\Omega) \to 0$, and the solution is called “a large solution” or “an explosive solution”, Ω is a bounded domain with smooth boundary in $\mathbf{R^N}$ $(N \geq 2)$, $m > 1$. The functions b and f satisfy

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(b_1) $b \in C^{\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$, is positive.
(f_1) $f \in C^1[0,+\infty), f(0) = 0, f$ is increasing on $[0,+\infty)$;
(f_2) $\int_1^{\infty} \frac{dv}{f^{\frac{1}{m-1}}(v)} < \infty$;

(f_3) there exists $C_f > 0$ such that

$$\lim_{s \to +\infty} f^{\frac{1}{m-1}-1}(s) f'(s) \int_s^{\infty} \frac{dv}{f^{\frac{1}{m-1}}(v)} = C_f.$$

(f_4) f satisfies Keller-Osserman condition

$$\int_1^{\infty} (\int_0^u f(s)ds)^{-1/m} du < \infty.$$

Quasilinear elliptic problems with boundary blow-up

$$\begin{cases} \text{div}(|\nabla u|^{m-2} \nabla u) = f(u(x)), & x \in \Omega, \\ u|_{\partial\Omega} = \infty, \end{cases} \tag{1.2}$$

have been studied, see [1-4] and the references therein. Diaz and Letelier [1] proved the existence and uniqueness of large solutions to the problem(1.2) both for $f(u) = u^{\gamma}, \gamma > m-1$(super-linear case) and $\partial\Omega$ being of the class C^2. Lu, Yang and E.H.Twizell [2] proved the existence of Large solutions to the problem(1.1) both for $f(u) = u^{\gamma}, \gamma > m-1, \Omega = \mathbf{R}^N$ or Ω being a bounded domain (super-linear case) and $\gamma \leq m-1, \Omega = \mathbf{R}^N$(sub-linear case) respectively.Quasilinear elliptic equation (system) with Dirichlet problem and other problem has been studied, see $[1-15]$.

When $m = 2$, problems (1.1) becomes

$$\Delta u = b(x)g(u), \quad x \in \Omega, \quad u|_{\partial\Omega} = +\infty. \tag{1.3}$$

The problem (1.3) arises from many branches of mathematics and applied mathematics, and have been discussed by many authors and in many contexts.see[16-36].

Now we introduce a class of functions. Let Λ denote the set of all positive nondecreasing functions in $C^1(0,\delta_0)(\delta_0 > 0)$ which satisfy

$$\lim_{t \to 0^+} \frac{d}{dt}(\frac{K(t)}{k(t)}) := C_k \in [0,\infty), \quad K(t) = \int_0^t k(s)ds. \tag{1.4}$$

The set Λ was first introduced by Cîstea and Rădulescu [23] for studying the boundary behavior and uniqueness of solutions of problem (1.3).

In this paper, we will investigate the exact asymptotic behavior of solutions near the boundary for the problem (1.1), when $m = 2$, Zhijun Zhang [37] have studied the boundary behavior of solutions of problem (1.3) for more general nonlinearities f. Our result generalize and improve the corresponding result of [37] in some sense. In the second section, we will give some preliminaries for the main result, in the last section, we will give the proof of the main result.

Our main result is summarized in the following theorem.

Theorem 1.1. Let f satisfy (f_1),(f_2),(f_3) and b satisfy (b_1) and (b_2), where (b_2):

$$\lim_{d(x)\to 0} \frac{(-1)^m b(x)}{K^{m-2}(d(x))k^m(d(x))} = b_0. \tag{1.5}$$

If

$$2C_f + (m-1)C_k > 2(m-1), \tag{1.6}$$

then for any solution u of problem (1.1) satisfies

$$\lim_{d(x)\to 0} \frac{u(x)}{\psi(\tau_0 K^2(d(x)))} = 1, \tag{1.7}$$

where ψ is uniquely determined by

$$\int_{\psi(t)}^{\infty} \frac{ds}{f^{\frac{1}{m-1}}(s)} = t, \quad \forall\ t > 0, \tag{1.8}$$

and

$$\tau_0 = \frac{1}{2}\left(\frac{b_0}{2C_f + (m-1)C_k - 2(m-1)}\right)^{\frac{1}{m-1}}.$$

2 Preliminaries

In this section, we present some bases of the theory which come from Senta [38], Preliminaries in Resnick [39], Introductions and the appendix in Maric [40].

Definition 2.1. A positive measurable function f defined on $[a, +\infty)$, for some $a > 0$, is called **regularly varying at infinity** with index ρ, written as $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbf{R}$,

$$\lim_{s\to\infty} \frac{f(\xi s)}{f(s)} = \xi^\rho. \tag{2.1}$$

In particular, when $\rho = 0$, f is called **slowly varying at infinity**.

Definition 2.2. A positive measurable function f defined on $[a, +\infty)$, for some $a > 0$, is called **rapidly varying at infinity** if for each $p > 1$

$$\lim_{s\to\infty} \frac{f(s)}{s^p} = \infty. \tag{2.2}$$

Clearly, if $f \in RV_\rho$, then $L(s) := f(s)/s^\rho$ is slowly varying at infinity.

Proposition 2.1 (Uniform convergence theorem). If $f \in RV_\rho$,then (2.1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$. Moreover, if $\rho < 0$, then uniform convergence holds on intervals of the form (a_1, ∞) with $a_1 > 0$; if $\rho > 0$, then uniform convergence holds on intervals $(a_1, \infty]$ provided f is bounded on $(a_1, \infty]$ for all $a_1 > 0$.

Proposition 2.2 (Representation theorem). A function L is slowly varying at infinity if and only if it may be written in the form

$$L(s) = \varphi(s) \exp(\int_{a_1}^{s} \frac{y(\tau)}{\tau} d\tau), \quad s \geq a_1, \tag{2.3}$$

for some $a_1 > a$, where the functions φ and y are measurable and for $s \to \infty, y(s) \to 0$, and$\varphi(s) \to c_0$,with $c_0 > 0$.

We call that

$$\hat{L}(s) = c_0 \exp(\int_{a_1}^{s} \frac{y(\tau)}{\tau} d\tau), \quad s \geq a_1, \tag{2.4}$$

is normalized slowly varying at infinity and

$$f(s) = c_0 s^\rho \hat{L}(s), \quad s \geq a_1, \tag{2.5}$$

is normalized regularly varying at infinity with index ρ (and written as $f \in NRV_\rho$).

Similarly, g is called normalized regularly varying at zero with index ρ ,written as $g \in NRVZ_\rho$ if $t \to g(1/t)$ belongs to NRV_ρ. A function $f \in RV_\rho$ belongs to NRV_ρ if and only if

$$f \in C^1[a_1, \infty), \text{ for some } a_1 > 0, \text{ and } \lim_{s \to \infty} \frac{sf'(s)}{f(s)} = \rho. \tag{2.6}$$

Proposition 2.3. If functions L, L_1 are slowly varying at infinity, then

(i) L^σ for every $\sigma \in \mathbf{R}$, $c_1 L + c_2 L_1$ $(c_1 \geq 0, c_2 \geq 0 \text{with} c_1 + c_2 > 0)$, $L \circ L_1$ (if $L_1(t) \to +\infty$ as $t \to +\infty$), are also slowly varying at infinity;

(ii) for every $\theta > 0$ and $t \to +\infty$,$t^\theta L(t) \to +\infty$ and $t^{-\theta} L(t) \to 0$;

(iii) for $\rho \in \mathbf{R}$ and $t \to +\infty$, $\frac{\ln(L(t))}{\ln t} \to 0$ and $\frac{\ln(t^\rho L(t))}{\ln t} \to \rho$.

Proposition 2.4. (Asymptotic behavior). If a function L is slowly varying at infinity, then for $a > 0$ and $t \to \infty$,

(i) $\int_a^t s^\beta L(s) ds \cong (\beta + 1)^{-1} t^{1+\beta} L(t)$, for $\beta > -1$;

(ii) $\int_t^\infty s^\beta L(s) ds \cong (-\beta - 1)^{-1} t^{1+\beta} L(t)$, for $\beta < -1$.

Proposition 2.5 (Asymptotic behavior). If a function H is slowly varying at zero, then for $a > 0$ and $t \to 0^+$,

(i) $\int_a^t s^\beta H(s) ds \cong (\beta + 1)^{-1} t^{1+\beta} H(t)$, for $\beta > -1$;

(ii) $\int_t^\infty s^\beta H(s) ds \cong (-\beta - 1)^{-1} t^{1+\beta} H(t)$, for $\beta < -1$.

Our results in this section are summarized in the following.

Lemma 2.1. If f satisfies (f_1), (f_2) and (f_3), then
(i) $C_f \in [1, +\infty)$;
(ii) If (f_3) holds for $C_f > 1$, then $f \in NRV_{C_f/(C_f-1)}$;
(iii) $C_f = 1$, f is rapidly varying at infinity.

Proof (i) Let

$$J(s) = f^{\frac{1}{m-1}-1}(s) f'(s) \int_s^{\infty} \frac{dv}{f^{\frac{1}{m-1}-1}(v)}, \forall \quad s > 0.$$

Integrating $J(s)$ from $a(a > 0)$ to t and integrate by parts, we obtain

$$\int_a^t J(s)ds = f^{\frac{1}{m-1}-1}(t) \int_t^{\infty} \frac{dv}{f^{\frac{1}{m-1}-1}(v)} - f^{\frac{1}{m-1}-1}(a) \int_a^{\infty} \frac{dv}{f^{\frac{1}{m-1}-1}(v)} + t - a, \forall \quad t > a.$$

It follows from the l'Hospital's rule that

$$0 \le \lim_{t\to\infty} \frac{f^{\frac{1}{m-1}-1}(t) \int_t^{\infty} \frac{dv}{f^{\frac{1}{m-1}-1}(v)}}{t} = \lim_{t\to\infty} \frac{1}{t} \int_a^t J(s)ds - 1 = \lim_{t\to\infty} J(t) - 1 = C_f - 1,$$

i.e.,$C_f \ge 1$.

(ii) By (i), we see that

$$\begin{aligned}
\lim_{s\to+\infty} \frac{f(s)}{sf'(s)} &= \lim_{s\to+\infty} \frac{f^{\frac{1}{m-1}}(s) \int_s^{\infty} \frac{dv}{f^{\frac{1}{m-1}}(v)}}{sJ(s)} \\
&= \frac{1}{C_f} \lim_{s\to+\infty} \frac{f^{\frac{1}{m-1}}(s) \int_s^{\infty} \frac{dv}{f^{\frac{1}{m-1}}(v)}}{s} \\
&= \frac{C_f - 1}{C_f}.
\end{aligned}$$

i.e.,$f \in NRV_{C_f/(C_f-1)}$ for $C_f > 1$.

(iii) When $C_f = 1$, we see by the proof of (iv) that

$$\lim_{s\to+\infty} \frac{f(s)}{sf'(s)} = 0.$$

Consequently, for arbitrary $p > 1$, there exists $S_0 > 0$ such that

$$\frac{f'(s)}{f(s)} > (p+1)s^{-1}, \quad \forall\, s \ge S_0,$$

Integrating the above inequality from S_0 to s, we obtain

$$\ln(f(s)) - \ln(f(S_0)) > (p+1)(\ln s - \ln S_0), \quad \forall\, s \ge S_0,$$

letting $s \to +\infty$, we see by Definition 2.2 that f is rapidly varying at infinity.

Lemma 2.2. Let f satisfy (f_1), (f_2),(f_3) and let ψ be the solution to the problem

$$\int_{\psi(t)}^{\infty} \frac{ds}{f^{\frac{1}{m-1}}(s)} = t, \quad \forall \ t > 0.$$

Then

(i) $-\psi'(t) = f^{\frac{1}{m-1}}(\psi(t)),\ \psi(t) > 0,\ t > 0,\ \psi(0) := \lim_{t \to 0^+} \psi(t) = +\infty$, and

$$\psi''(t) = \frac{1}{m-1} f^{\frac{2}{m-1}-1}(\psi(t)) f'(\psi(t)), \quad t > 0;$$

(ii) $\psi \in NRVZ_{-(C_f-1)}$;

(iii) $-\psi' = f^{\frac{1}{m-1}} \circ \psi \in NRVZ_{-C_f/(m-1)}$;

Proof. By the definition of ψ and a direct calculation, we show that (i) holds. (ii) It follows from the proof of Lemma 2.1 that

$$\lim_{t \to 0^+} \frac{t\psi'(t)}{\psi(t)} = -\lim_{t \to 0^+} \frac{t f^{\frac{1}{m-1}}(\psi(t))}{\psi(t)} = -\lim_{s \to +\infty} \frac{f^{\frac{1}{m-1}}(s) \int_s^{\infty} \frac{dv}{f^{\frac{1}{m-1}}(v)}}{s} = -(C_f - 1),$$

i.e., $\psi \in NRVZ_{-(C_f-1)}$.

(iii) (f_3) implies

$$\begin{aligned} \lim_{t \to 0^+} \tfrac{t\psi''(t)}{\psi'(t)} &= -\lim_{t \to 0^+} \tfrac{t}{m-1} f^{\frac{1}{m-1}-1}(\psi(t)) f'(\psi(t)) \\ &= -\lim_{s \to +\infty} \tfrac{1}{m-1} f^{\frac{1}{m-1}-1}(s) f'(s) \int_s^{\infty} \tfrac{dv}{f^{\frac{1}{m-1}}(v)} \\ &= -C_f/(m-1). \end{aligned}$$

Lemma 2.3. $k \in \Lambda$ implies:

(i) $\lim_{t \to 0^+} \frac{K(t)}{k(t)} = 0$;

(ii) $C_k \in [0, 1]$ and $\lim_{t \to 0^+} \frac{K(t)k'(t)}{k^2(t)} = 1 - C_k$.

3 Proofs of the main result

Before prove our main results,we give the following Lemma 3.1(From [2,4]).

Lemma 3.1. (weak comparison principle) Let Ω be a bounded domain in $\mathbf{R}^{\mathbf{N}}$ $(\mathbf{N} \geq \mathbf{2})$ with smooth boundary $\partial\Omega$ and $\varphi :\ (0, a) \to (0, a)$ be continuous and non-decreasing, let $u_1,\ u_2 \in W^{1,m}(\Omega)$ satisfy

$$\int_{\Omega} |\nabla u_1|^{m-2} \nabla u_1 \nabla \psi dx + \int_{\Omega} \varphi u_1 \psi dx \leq \int_{\Omega} |\nabla u_2|^{m-2} \nabla u_2 \nabla \psi dx + \int_{\Omega} \varphi u_2 \psi dx,$$

For all non-negative $\psi \in W_0^{1,m}(\Omega)$. Then the inequality

$$u_1 \le u_2, \quad \text{on } \partial\Omega,$$

implies that

$$u_1 \le u_2, \quad \text{in } \Omega.$$

Now let $v_0 \in C^{2+\alpha} \cap C^1(\bar{\Omega})$ be the unique solution of the problem

$$\text{div}(|\nabla v_0|^{m-2}\nabla v_0) = 1, \quad v_0 > 0, \ \ x \in \Omega, \quad v_0|_{\partial\Omega} = 0. \tag{3.1}$$

By the Höpf maximum principle in [41], we see that

$$\nabla v_0 \neq 0, \ \ \forall\, x \in \partial\Omega, \quad \text{and} \quad c_1 d(x) \le v_0 \le c_2 d(x), \ \ \forall\, x \in \Omega, \tag{3.2}$$

where c_1, c_2 are positive constants. For any $\delta > 0$, we define

$$\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\}.$$

Since Ω is smooth, there exists $\delta_0 > 0$ such that $d \in C^2(\Omega_{\delta_0})$ and

$$|\nabla d(x)| = 1. \tag{3.3}$$

Proof of Theorem 1.1. Let $\varepsilon \in (0, \sqrt[m-1]{b_0}/4)$ and

$$\tau_1 = \tau_0 - \frac{2\varepsilon\tau_0}{\sqrt[m-1]{b_0}}, \quad \tau_2 = \tau_0 + \frac{2\varepsilon\tau_0}{\sqrt[m-1]{b_0}}.$$

It follows that

$$\tau_0/2 < \tau_1 < \tau_0 < \tau_2 < 2\tau_0.$$

By $(b_1), (b_2)$ and Lemma 2.1-2.3, we see that there is $\delta_\varepsilon \in (0, \delta_0/2)$(which is corresponding to ε) sufficiently small such that

(r_1) $(b_0 - \varepsilon)k^2(d(x) - \rho) \le (b_0 - \varepsilon)k^2(d(x)) < b(x), \ \ x \in D_\rho^- = \Omega_{2\delta_\varepsilon}/\bar{\Omega}_\rho;$

$b(x) < (b_0+\varepsilon)k^2(d(x)) \le (b_0+\varepsilon)k^2(d(x)+\rho), \ \ x \in D_\rho^+ = \Omega_{2\delta_\varepsilon - \rho},$ where $\rho \in (0, \delta_\varepsilon)$.

(r_2) For i=1,2,

$$4(2\tau_0)^{m-1}|(m-1)\tau_i K^2(t) f^{\frac{1}{m-1}-1}(\psi(\tau_i K^2(t)))f'(\psi(\tau_i K^2(t))) - C_f| + 2(2\tau_0)^{m-1}$$
$$(m-1)|\tfrac{k'(t)K(t)}{k^2(t)} - (1 - C_k)| + 2(2\tau_0)^{m-1}\tfrac{K(t)}{k(t)}|\triangle d(x)| < \varepsilon, \ \ \forall (x,t) \in \Omega_{2\delta_\varepsilon} \times (0, 2\delta_\varepsilon).$$

Let

$$d_1(x) = d(x) - \rho, \quad d_2(x) = d(x) + \rho, \tag{3.4}$$

$$\bar{u}_\varepsilon = \psi(\tau_1 K^2(d_1(x))), \quad x \in D_\rho^- \quad \text{and} \quad \underline{u}_\varepsilon = \psi(\tau_2 K^2(d_2(x))) \quad x \in D_\rho^+. \tag{3.5}$$

It follows that, for $x \in D_\rho^-$

$$\begin{aligned}
&\text{div}(|\nabla u|^{m-2}\nabla u) - b(x)f(\bar{u}_\varepsilon(x)) \\
&= (2\tau_1)^{m-1}[(m-1)(\psi'(\tau_1 K^2(d_1(x))))^{m-2}\psi''(\tau_1 K^2(d_1(x)))2\tau_1 K^m(d_1(x))k^m(d_1(x)) \\
&\quad +(m-1)(\psi'(\tau_1 K^2(d_1(x))))^{m-1}K^{m-2}(d_1(x))k^m(d_1(x)) \\
&\quad +(m-1)(\psi'(\tau_1 K^2(d_1(x))))^{m-1}K^{m-1}(d_1(x))k^{m-2}(d_1(x))k'(d_1(x)) \\
&\quad +(\psi'(\tau_1 K^2(d_1(x))))^{m-1}K^{m-1}(d_1(x))k^{m-1}(d_1(x))\triangle d(x)] - b(x)f(\tau_1 K^2(d_1(x)))) \\
&= (-1)^m f(\psi(\tau_1 K^2(d_1(x))))k^m(d_1(x))K^{m-2}(d_1(x))[2^m{\tau_1}^{m-1}((m-1)\tau_1 K^2(d_1(x)) \\
&\qquad f^{\frac{1}{m-1}-1}(\psi(\tau_1 K^2(d_1(x))))f'(\psi(\tau_1 K^2(d_1(x)))) - C_f) + 2^m{\tau_1}^{m-1}C_f - (m-1)(2\tau_1)^{m-1} \\
&\quad -(m-1)(2\tau_1)^{m-1}(\tfrac{k(d_1(x))k'(d_1(x))}{k^2(d_1(x))} - (1-C_k)) - (m-1)(2\tau_1)^{m-1}(1-C_k) \\
&\quad -(2\tau_1)^{m-1}\tfrac{K(d_1(x))}{k(d_1(x))}\triangle d(x) - (\tfrac{(-1)^m b(x)}{K^{m-2}(d_1(x))k^m(d_1(x))} - b_0) - b_0] \\
&\le |(-1)^m f(\psi(\tau_1 K^2(d_1(x))))k^m(d_1(x))K^{m-2}(d_1(x))|\{(2\tau_1)^{m-1}[2|(m-1)\tau_1 K^2(d_1(x)) \\
&\quad f^{\frac{1}{m-1}-1}(\psi(\tau_1 K^2(d_1(x))))f'(\psi(\tau_1 K^2(d_1(x)))) - C_f| + (m-1)|\tfrac{k'(d_1(x))K(d_1(x))}{k^2(d_1(x))} - (1-C_k)| \\
&\quad +\tfrac{K(d_1(x))}{k(d_1(x))}|\triangle d(x)|] + 2^m{\tau_1}^{m-1}C_f - (m-1)(2\tau_1)^{m-1}(m-1)(2\tau_1)^{m-1}(1-C_k) \\
&\quad -(\tfrac{(-1)^m b(x)}{K^{m-2}(d_1(x))k^m(d_1(x))} - b_0) - b_0\} \\
&\le (2\tau_1)^{m-1}\tfrac{\varepsilon}{2(2\tau_0)^{m-1}} - (\tfrac{(-1)^m b(x)}{K^{m-2}(d_1(x))k^m(d_1(x))} - b_0) \\
&\quad +(2\tau_1)^{m-1}[2C_f + (m-1)C_k - 2(m-1)] - b_0 \\
&\le \tfrac{\varepsilon}{2} - \tfrac{\varepsilon}{2} + \tfrac{(2\tau_1)^{m-1}}{(2\tau_0)^{m-1}}b_0 - b_0 \le 0.
\end{aligned}$$

i.e., $\bar{u}_\varepsilon$ is a supersolution of Eq.(1.1) in D_ρ^-.

In a similar way, for $x \in D_\rho^+$

$$\begin{aligned}
&\text{div}(|\nabla u|^{m-2}\nabla u) - b(x)f(\underline{u}_\varepsilon(x)) \\
&= (2\tau_2)^{m-1}[(m-1)(\psi'(\tau_2 K^2(d_2(x))))^{m-2}\psi''(\tau_2 K^2(d_2(x)))2\tau_2 K^m(d_2(x))k^m(d_2(x)) \\
&\quad +(m-1)(\psi'(\tau_2 K^2(d_2(x))))^{m-1}K^{m-2}(d_2(x))k^m(d_2(x)) \\
&\quad +(m-1)(\psi'(\tau_2 K^2(d_2(x))))^{m-1}K^{m-1}(d_2(x))k^{m-2}(d_2(x))k'(d_2(x)) \\
&\quad +(\psi'(\tau_2 K^2(d_2(x))))^{m-1}K^{m-1}(d_2(x))k^{m-1}(d_2(x))\triangle d(x)] - b(x)f(\tau_2 K^2(d_2(x)))) \\
&= (-1)^m f(\psi(\tau_2 K^2(d_2(x))))k^m(d_2(x))K^{m-2}(d_2(x))[2^m{\tau_2}^{m-1}((m-1)\tau_2 K^2(d_2(x)) \\
&\qquad f^{\frac{1}{m-1}-1}(\psi(\tau_2 K^2(d_2(x))))f'(\psi(\tau_2 K^2(d_2(x)))) - C_f) + 2^m{\tau_2}^{m-1}C_f - (m-1)(2\tau_2)^{m-1} \\
&\quad -(m-1)(2\tau_2)^{m-1}(\tfrac{k(d_2(x))k'(d_2(x))}{k^2(d_2(x))} - (1-C_k)) - (m-1)(2\tau_2)^{m-1}(1-C_k) \\
&\quad -(2\tau_2)^{m-1}\tfrac{K(d_2(x))}{k(d_2(x))}\triangle d(x) - (\tfrac{(-1)^m b(x)}{K^{m-2}(d_2(x))k^m(d_2(x))} - b_0) - b_0] \\
&\ge |(-1)^m f(\psi(\tau_2 K^2(d_2(x))))k^m(d_2(x))K^{m-2}(d_2(x))|\{(2\tau_2)^{m-1}[-2|(m-1)\tau_2 K^2(d_2(x)) \\
&\quad f^{\frac{1}{m-1}-1}(\psi(\tau_2 K^2(d_2(x))))f'(\psi(\tau_2 K^2(d_2(x)))) - C_f| - (m-1)|\tfrac{k'(d_2(x))K(d_2(x))}{k^2(d_2(x))} - (1-C_k)| \\
&\quad -\tfrac{K(d_2(x))}{k(d_2(x))}|\triangle d(x)|] + 2^m{\tau_2}^{m-1}C_f - (m-1)(2\tau_2)^{m-1}(m-1)(2\tau_2)^{m-1}(1-C_k) \\
&\quad -(\tfrac{(-1)^m b(x)}{K^{m-2}(d_2(x))k^m(d_2(x))} - b_0) - b_0\} \\
&\ge -(2\tau_2)^{m-1}\tfrac{\varepsilon}{2(2\tau_0)^{m-1}} - (\tfrac{(-1)^m b(x)}{K^{m-2}(d_2(x))k^m(d_2(x))} - b_0) \\
&\quad +(2\tau_2)^{m-1}[2C_f + (m-1)C_k - 2(m-1)] - b_0 \\
&\ge -\tfrac{\varepsilon}{2} + \varepsilon + \tfrac{(2\tau_2)^{m-1}}{(2\tau_0)^{m-1}}b_0 - b_0 \ge 0.
\end{aligned}$$

We can show that $\underline{u}_\varepsilon$ is a subsolution of of Eq.(1.1) in D_ρ^+. Now let u be an arbitrary solution of problem (1.1), we assert that there exists a positive constant M such that

$$u \leq Mv_0(x) + \bar{u}_\varepsilon, \quad x \in D_\rho^-, \tag{3.6}$$

$$\underline{u}_\varepsilon \leq u + Mv_0(x), \quad x \in D_\rho^+, \tag{3.7}$$

where v_0 is the solution of problem (3.1). In fact, we may choose a large M such that

$$u \leq Mv_0(x) + \bar{u}_\varepsilon, \quad \text{on} \quad \Gamma_{2\delta_\varepsilon} := \{x \in \Omega : d(x) = 2\delta_\varepsilon\}.$$

By (f_1), we see that $Mv_0(x) + \bar{u}_\varepsilon$ is also a supersolution of Eq.(1.1) in D_ρ^-. Since $u < \bar{u}_\varepsilon$ on $\Gamma_\rho := \{x \in \Omega : d(x) = \rho\}$, (3.6) follows by Lemma 3.1. In a similar way, we can show (3.7). Hence, $x \in D_\rho^- \cap D_\rho^+$, by letting $\rho \to 0$, we have

$$1 - \frac{Mv_0(x)}{\psi(\tau_2 K^2(d_2(x)))} \leq \frac{u(x)}{\psi(\tau_2 K^2(d_2(x)))}$$

and

$$\frac{u(x)}{\psi(\tau_1 K^2(d_2(x)))} \leq 1 + \frac{Mv_0(x)}{\psi(\tau_1 K^2(d_2(x)))}$$

Consequently,

$$1 \leq \lim_{d(x)\to 0} \inf \frac{u(x)}{\psi(\tau_2 K^2(d_2(x)))} \quad \text{and} \quad \lim_{d(x)\to 0} \sup \frac{u(x)}{\psi(\tau_1 K^2(d_2(x)))} \leq 1.$$

Thus by letting $\varepsilon \to 0$, we have

$$\lim_{d(x)\to 0} \frac{u(x)}{\psi(\tau_0 K^2(d_2(x)))} = 1.$$

The proof is finished.

The existence of solutions to problem (1.1) is similar as that in [4], so we omit it in this article.

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Competing Interests

The authors declare that no competing interests exist.

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