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On Mono-correct Modules

Anta Niane GUEYE[1] **and Cheikh Thiécoumba GUEYE**[*1]

[1] *Laboratoire d'Algèbre, de Cryptologie, de Géométrie Algèbrique et Applications (LACGAA), Département de Mathématiques et Informatique, Faculté des Sciences et Techniques, Université Cheikh Anta Diop de DAKAR, SENEGAL*

Research Article

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Abstract

Let R be a commutative ring. It is well known that any artinian module is co-hopfian and any artinian module is mono-correct, but the converse is not true. Furthermore, commutative rings on which co-hopfian modules are artinian have been characterized. The aim of this work is to study the existence of commutative rings R on which mono-correct R-modules are artinian.
We establish that if there exists a commutative ring on which mono-correct R-modules are artinian, then it is an artinian ideal principal one. And on a non-zero commutative artinian principal ideal ring R, we have shown the existence of R-modules which are mono-correct but are not artinian.
Hence a non-singleton unital commutative ring R such that every mono-correct R-module is artinian does not exist.

Keywords: Mono-correct module; artinian; co-hopfian; artinian principal ideal ring
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1 Introduction

It is well known that any artinian module is mono-correct, but the converse is not true. $\mathbb{Z}$ considered as a $\mathbb{Z}$-module is mono-correct but is not artinian. We recall that any artinian R-module is co-hopfian but a co-hopfian R-module is not necessarily artinian. Several studies have been done on co-hopfian modules and on rings on which co-hopfian modules verify some interesting properties [see [7], [9], [1], [5]]. Mono-correctness of modules has been studied in [8], and in [10] it is shown that, R being a ring, for an R-module M, the class $\sigma[M]$ of all M-subgenerated modules is mono-correct if and only if M is semisimple. In [4], commutative rings on which any finitely generated module is mono-correct have been characterized.
The motivation of our investigation is the well-known characterization of rings on which co-hopfian

E-mail: cheikht.gueye@.ucad.edu.sn

modules are artinian [6]. By analogy, we are led to study the existence of commutative rings on which mono-correct modules are artinian. First, we show that if such a ring exists, then it is an artinian principal ideal one. After that, we prove on a non-zero artinian principal ideal ring the existence of modules which are mono-correct but are not artinian. Then we have established that there does not exist any commutative ring with identity $1 \neq 0$ on which mono-correct modules are artinian.

2 Definitions and Preliminaries

For the sake of self-containedness and the convenience of the reader, we recall in this section the main definitions and preliminaries we shall need to establish the main results. We denote by R-MOD the category of all R-modules.

Definition 2.1. Two modules M and N are called mono-equivalent if there are monomorphisms

$$f : M \longrightarrow N \text{ and } g : N \longrightarrow M.$$

We denote $M \overset{m}{\simeq} N$.

Definition 2.2. Two modules M and N are called equivalent if there exists an isomorphism

$$h : M \longrightarrow N.$$

We denote $M \simeq N$.

Definition 2.3. An R-module M is said to be mono-correct if for any R-module N,

$$M \overset{m}{\simeq} N \text{ implies } M \simeq N.$$

Proposition 2.1. *$\mathbb{Z}$ as a $\mathbb{Z}$-module is mono-correct.*

Proof. In fact if N is a $\mathbb{Z}$-module, f and g two monomorphisms $f : \mathbb{Z} \longrightarrow N$ and $g : N \longrightarrow \mathbb{Z}$, we have $N \simeq g(N)$ and $g(N)$ is a $\mathbb{Z}$ submodule. Therefore there exists $n \in \mathbb{Z}$ such that $g(N) = n\mathbb{Z}$. Thus $\mathbb{Z} \simeq n\mathbb{Z} = g(N) \simeq N$. Hence $\mathbb{Z}$ is mono-correct. $\square$

We recall that $\mathbb{Z}$ is not artinian.

Definition 2.4. A class $\mathcal{C}$ of objects in a category $\mathfrak{C}$ is said to be mono-correct if for any $A, B \in \mathcal{C}$, $A \overset{m}{\simeq} B$ implies $A \simeq B$.

Definition 2.5. An R-module M is said to be co-hopfian if every injective endomorphism $f : M \to M$ is an automorphism.

Example 2.1. *Any artinian module is co-hopfian.*

Proposition 2.2. *For a commutative ring R, any co-hopfian module is mono-correct.*

Proof. Let R be a commutative ring and M a co-hopfian R-module. Let N be an R-module. If there are monomorphisms $f : M \longrightarrow N$ and $g : N \longrightarrow M$, then $g \circ f : M \to M$ is an injective endomorphism. Hence $g \circ f$ is an automorphism, therefore g is surjective. This implies that $M \simeq N$, thus M is mono-correct. $\square$

Definition 2.6. A submodule H of an R-module M is said to be fully invariant in M if for any R-endomorphism f of M, we have $f(H) \subset H$.

Proposition 2.3. *Let M be a direct sum of submodules H_j $(j \in J)$. If for all j, H_j is co-hopfian and fully invariant in M, then M is co-hopfian.*

Proof. Assume that for every $j \in J$, H_j is co-hopfian and fully invariant in M. Let f be an injective endomorphism of M. The restriction f_j of f to H_j is an automorphism. Since M is a direct sum of the H_j's $(j \in J)$, then f is bijective, hence M is co-hopfian. □

Definition 2.7. A ring R is said to be an I-Ring if any co-hopfian R-module is artinian.

The following proposition gives a characterization of commutative I-rings.

Proposition 2.4. *[6] Let R be a commutative ring. Then the following assertions are equivalent:*

1. *R is an I-Ring;*
2. *R is an artinian principal ideal ring;*
3. *Every R-module is a direct sum of cyclic submodules.*

We have also

Lemma 2.2. *[6] Let $R = \prod_{j \in J} R_j$. Then R is an I-Ring if and only if J is finite and for all $j \in J$, R_j is an I-Ring.*

We shall need also

Proposition 2.5. *[9] For a commutative ring R, the following assertions are equivalent:*

1. *Any injective endomorphism of a finitely generated R-module is an isomorphism.*
2. *Every prime ideal of R is maximal.*

Now we are in a position to establish the following

Proposition 2.6. *Let R be an I-Ring and M an R-module. If every direct summand of M is fully invariant in M, then M is artinian.*

Proof. Let M be an R-module, then by Proposition (2.4) $M = \bigoplus_{j \in J} M_j$, where M_j are cyclic submodules, and thus finitely generated. R is an I-Ring implies that every prime ideal of R is maximal, hence for all $j \in J$, M_j is co-hopfian by Proposition (2.5). $M = \bigoplus_{i \in J} M_j$ and the M_j's are fully invariant in M, so it follows that M is co-hopfian by Proposition (2.3), and since R is an I-Ring, M is artinian. □

We need the following

Definition 2.8. Let M be an R-module. An R-module P is said to be generated by M or M-generated if, for every pair of distinct morphisms $f, g : P \longrightarrow Q$, $Q \in R$-MOD, there is a morphism $h : M \longrightarrow P$ and $hf \neq hg$.

Definition 2.9. Let M be an R-module. An R-module N is said to be subgenerated by M if N is isomorphic to a submodule of an M-generated module. We let $\sigma[M]$ denote the full subcategory of R-MOD whose objects are all R-modules subgenerated by M.

Proposition 2.7. *[11] Let M be an R-module. Then for $N \in \sigma[M]$, all factor modules and submodules of N belong to $\sigma[M]$.*

Proposition 2.8. *[10] For a module M, the following assertions are equivalent:*

1. *The class of all modules in $\sigma[M]$ is mono-correct.*
2. *Every module in $\sigma[M]$ is mono-correct.*
3. *M is semisimple.*

3 The main results

Let R be a commutative ring with identity $1 \neq 0$. Assume that R has the property that any mono-correct R-module is artinian. In the sequel, such a ring will be called an (M)-Ring.

Example 3.1. *If an I-Ring R is such that any direct summand of an R-module M is fully invariant in M, then it is an (M)-Ring by Proposition (2.6).*

Proposition 3.1. *If R is an (M)-Ring, then R is an artinian principal ideal ring.*

To establish the proof, we need the following

Lemma 3.2. *Let R be an (M)-Ring, then R is artinian.*

Proof. Assume that R is an (M)-Ring. Let K be the total ring of fractions of R. Then K is an R-module. Let us show that K is co-hopfian.
Let f be an injective R-endomorphism of K. For every $x \in K$, $x = s^{-1}a$ where $s \in R$, $a \in R$ and $s \neq 0$, we have $sf(x) = sf(s^{-1}a) = f(ss^{-1}a) = f(a) = af(1)$. Therefore $f(x) = xf(1)$, then f is an automorphism, hence K is mono-correct. It follows that K is artinian and then R is also artinian. □

Lemma 3.3. *Every homomorphic image of an (M)-Ring is an (M)-Ring.*

Proof. Let A be an (M)-Ring, $\varphi : A \longrightarrow B$ a surjective ring homomorphism, and M a mono-correct B-module. The following map:

$$\begin{aligned} A \times M &\longrightarrow M \\ (a, m) &\longmapsto \varphi(a)m = am \end{aligned}$$

induces an A-module structure on the additive abelian group M. Let us show that M is a mono-correct A-module. Let N be an A-module, $f : M \longrightarrow N$ and $g : N \longrightarrow M$ two A-monomorphisms. Let us establish that, N is a B-module. For all $b \in B$, for all $x \in N$, there exists $a \in A$ such that $\varphi(a) = b$. We consider the following product

$$b.x = ax \in N. \tag{3.1}$$

This product is well defined, since for all $a, a' \in A$ such that $\varphi(a) = \varphi(a')$ and for all $x \in N$, we have g injective implies that $g(N) \simeq N$ and

$$\varphi(a).g(x) = ag(x) = g(ax)$$

$$\varphi(a').g(x) = a'g(x) = g(a'x).$$

Then

$$g(ax) = g(a'x)$$

hence

$$ax = a'x.$$

By (3.1), N is a B-module. we have also that f and g are B-monomorphisms, M is a mono-correct B-module, then we deduce $M \simeq N$. This implies that M is a mono-correct A-module and then M is artinian. □

Lemma 3.4. *Let $R = \prod_{i=1}^{n} R_i$, then R is an (M)-Ring if and only if all R_i are (M)-Rings*

Proof. Assume that $R = \prod_{i=1}^{n} R_i$ and R is an (M)-Ring. Then the canonical projections $p_i : R \longrightarrow R_i$ $i \in \{1, 2, ..., n\}$ are surjective homomorphisms and by Lemma (3.3) all R_i are (M)-Rings.
Conversely, we assume that $R = \prod_{i=1}^{n} R_i$ and the R_i's are (M)-Rings. We want to show that R is an (M)-Ring. Let M be a mono-correct R-module, as $R = \prod_{i=1}^{n} R_i$ we can write $M = \bigoplus_{i=1}^{n} M_i$ with $M_i = Me_i$ where $e_i = (\delta_i^j)_{j=1}^n = (0, 0, .., \underset{\underset{i^{th} column}{\uparrow}}{1}, 0, ..., 0) \in R$, $1 \in R_i$ and for all $i \in \{1, 2, ..., n\}$, M_i is an R_i-module.
For all $i \in \{1, 2, ..., n\}$, let us show that M_i is a mono-correct R_i-module. If N_i is an R_i-module, $f_i : M_i \longrightarrow N_i$ and $g_i : N_i \longrightarrow M_i$ two monomorphisms, we have

$$f = \prod_{i=1}^{n} f_i : \bigoplus_{i=1}^{n} M_i \longrightarrow \bigoplus_{i=1}^{n} N_i \text{ and } g = \prod_{i=1}^{n} g_i : \bigoplus_{i=1}^{n} N_i \longrightarrow \bigoplus_{i=1}^{n} M_i$$

are R-monomorphisms. Therefore $\bigoplus_{i=1}^{n} M_i \simeq \bigoplus_{i=1}^{n} N_i$, then $M_i \simeq N_i$ for all $i \in \{1, 2, ..., n\}$. It follows that the M_i's are mono-correct. As the R_i's are (M)-Rings, we deduce that M_i is artinian for all $i \in \{1, 2, ..., n\}$, hence $M = \bigoplus_{i=1}^{n} M_i$ is artinian. □

Lemma 3.5. *Let R be a commutative artinian ring. If R has a non-principal ideal, then there exists a mono-correct R-module which is not artinian.*

Proof. It is known that R is a finite product of local artinian rings. Then we can assume that R is a local artinian ring with Jacobson radical $J(R) = aR + bR$ with the conditions $a^2 = b^2 = ab = 0$ and $a \neq 0, b \neq 0$. Then there exists by [3] a local artinian principal ideal subring C of R with Jacobson radical $J(R) = aC$ such that $R = C \oplus bC$ as C-modules. Let us consider the free C-module

$$M = \bigoplus_{i=0}^{\infty} Ce_i$$

with infinite countable basis $\{e_i, i \in \mathbb{N}\}$ and σ the endomorphism of C-modules defined as follows $\sigma(e_0) = 0$, and $\sigma(e_i) = ae_{i-1}$ for $i \geq 1$. Let Φ be the ring homomorphism:

$$\begin{aligned} \Phi : R = C \oplus bC &\longrightarrow End_C M \\ \alpha + b\lambda &\longrightarrow \alpha id_M + \lambda\sigma \end{aligned}$$

where id_M denotes the identity homomorphism of the C-module M. By [2], M has an R-module structure, and M is a co-hopfian R-module which is not finitely generated. As a co-hopfian R-module, M is mono-correct by Proposition (2.2). Since M is not a finitely generated R-module, M is not artinian. □

The proof of Proposition (3.1) is given by Lemma (3.2) and Lemma (3.5).
Now we are going to show that a non-zero commutative artinian principal ideal ring is not an (M)-Ring.

Proposition 3.2. *Let R be a non-zero commutative artinian principal ideal ring. Then there exists a mono-correct R-module which is not artinian.*

Proof. If R is an artinian principal ideal ring and $1 \neq 0$, then there exists $n \geq 1$ such that $R = \prod_{i=1}^{n} R_i$ where the R_i's are local artinian principal ideal rings. By Lemma (2.2) and Lemma (3.4), we can assume that R is a local artinian principal ideal ring. Let J be the unique maximal ideal of R. $S = R/J$ is a simple R-module and any S-module M is an R-module by the following product: for every $r \in R$, $x \in M$, $rx = \overline{r}x$ where $\overline{r} \in S$.
For r, $s \in R$ and x, $y \in M$, we have

- $r(x + y) = \overline{r}(x + y) = \overline{r}x + \overline{r}y = rx + ry$
- $(r + s)x = \overline{(r + s)}x = (\overline{r} + \overline{s})x = \overline{r}x + \overline{s}x = rx + sx$
- $r(sx) = \overline{r}(\overline{s}x) = \overline{rs}x = (rs)x$
- $1x = \overline{1}x = x.$

Let us consider the infinite countable S-vector space $V = S^{(\mathbb{N})}$. V is a semisimple R-module. S is a field and then V is mono-correct as an S-module. Let us show that V is mono-correct as an R-module. Let N be an R-module, $f : V \longrightarrow N$ and $g : N \longrightarrow V$ two R-monomorphisms. N is isomorphic to $g(N)$ and $g(N)$ is a submodule of V, then $g(N) \in \sigma[V]$ by Proposition (2.7). As V is semisimple, all modules in $\sigma[V]$ are mono-correct by Proposition (2.8), hence $g(N)$ is mono-correct. Let us consider:

$$\widetilde{f} : V \xrightarrow{f} N \xrightarrow{i} g(N)$$

and

$$\widetilde{g} : g(N) \xrightarrow{i} N \xrightarrow{g} V.$$

$\widetilde{f}$ and $\widetilde{g}$ are monomorphisms and $g(N)$ is mono-correct, thus $g(N) \simeq V$, and hence $N \simeq V$. This implies that V is mono-correct as an R-module. But V is not artinian since it is an infinite dimensional vector space. ☐

4 Conclusion

Artinian modules are co-hopfian and mono-correct, the converse is false. Commutative rings on which every co-hopfian module is artinian exist and have been characterized. By analogy we have studied and shown in this paper that a non-singleton unital commutative ring R such that every mono-correct module is artinian does not exist. In fact we have established that if such a ring exists then it is an artinian principal ideal one, and on a non-singleton unital artinian principal ideal ring R we have shown the existence of mono-correct R-modules which are not artinian.
Following this result on mono-correct modules, the authors think that finding non-artinian mono-correct R-module when R is not necessarily commutative can be very interesting. And knowing that any co-hopfian module is mono-correct, another opening problem is to find when a mono-correct module is co-hopfian or try to characterize the rings R on which every mono-correct module is co-hopfian.

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Competing Interests

The authors declare that no competing interests exist.

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